# Blockvorlesung : Black Holes

## SoSe 2022

Abhiram Kidambi (abhiram.kidambi@ipmu.jp)
Office : 208B, 16:00 - 18:00

Course Website :

https://abhirammk.github.io/bh_sose22

L1: Stellar collapse + derivation of Schwarzschild metric

L2: Schwarzschild spacetime

L3: Singularity theorems + initial value problem

L4: Penrose diagram + Reissner-Nordström

L5: Kerr Black Hole + Kerr-Newman

L6: Black Hole mechanics

L7: BH evaporation + Hawking radiation + information paradox

L8: Buffer, (BTZ black hole, BH orbits, Higher dim BH's....)

## Literature:

Physics: R. Wald – General Relativity

Hawking, Ellis – Large Scale structure of spacetime

S. Carroll – Spacetime and geometry

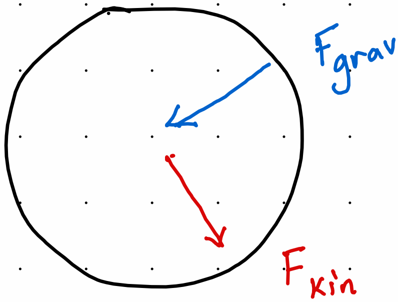
R. Wald – QFT in curved spacetime and BH thermodynamics

Sachs, Wu – General Relativity for mathematicians

## Lecture notes:

- Townsend's Black Holes notes (Part III – Based on Hawking's lectures)
- Matthias Blau's GR notes (available online)

# Stellar Collapse




At equilibrium, $F_{kin} = F_{grav}$ i.e. $E_{grav} + E_{kin}$ is minimized.

$$E_{grav} \sim -\frac{GM^2}{R} \quad , \quad E_{kin} \sim k_B T N$$

$N$ = # of particles in the star.

Recall $PV = nRT$, $n = \dfrac{N}{N_A}$

$$PV = \frac{NRT}{N_A} = N k_B T$$

$$\boxed{\begin{aligned} E_{kin} &\sim k_B T n R^3 \\ &\sim \langle E \rangle n R^3 \end{aligned}}$$

Eventually the nuclear processes in the star stops. This means that the star starts to collapse under gravity. Since the star starts to cool, we have to work in the idealistic limit that

$T \to 0K$. But $P \not\to 0$ as $T \to 0K$.

Why? Quantum degeneracy.

What is quantum degeneracy? The number density of particles is 1 per $(\text{Compton wavelength})^3$.

Compton wavelength of a particle with momentum p is the wavelength of a photon with the same energy as that particle.

$$\lambda_e = \frac{\hbar}{\langle p_e \rangle} \longrightarrow \text{Reduced Compton wavelength}$$

$$n_e = \lambda_e^{-3}$$

Note: Since $\lambda_c \propto \frac{1}{m}$, lower mass particles become quantum degenerate first.

We consider a non-relativistic electrongas

(Fermi energy $\frac{P_F^2}{m_e} \sim \frac{\langle P \rangle^2}{m_e} \ll m_e c^2$ )

Homework: Check this

A: $\lambda_e \sim 2.47 * 10^{-12} \longrightarrow \frac{\hbar}{\langle P_e \rangle}$

$$\langle P_e \rangle = \frac{1.054 * 10^{-34}}{\lambda_e} = \frac{1.054 \times 10^{-34}}{2.47 * 10^{-12}} \frac{Js}{m}$$

$$= 0.426 * 10^{-22} \ m/s$$

$$\langle P_e \rangle^2 = 0.181 * 10^{-44} \sim 1.8 * 10^{-45} \ m^2/s^2$$

$$(m_e c)^2 = (9.1 \times 10^{-31} * 3 \times 10^8)^2 = 745 * 10^{-46}$$

$$\frac{(m_e c)^2}{(P)^2} \sim \frac{7.45 * 10^{-44}}{1.8 * 10^{-45}} = 41.3$$

i.e the electron is only 0.02 % the relativistic limit.

Since electrons are non relativistic,

$\langle E \rangle \sim \frac{p_e^2}{m_e}$. At electron degeneracy, $n_e \gg n_p$. $\therefore$ we assume $n \sim n_e$.

$$E_{kin} \sim nR^3 \frac{p_e^2}{m_e} \sim \frac{n_e R^3 \hbar^2}{n_e^{-2/3} m_e} \sim \frac{n_e^{5/3} R^3 \hbar^2}{m_e}$$

$N \sim \frac{M}{m_p}$ (Star is electrically neutral)

$m_p \rightarrow$ mass of proton

$n_e \sim \frac{M}{m_p R^3}$. Plug back into $E_{kin}$.

$$E_{kin} \sim \frac{M^{5/3}}{m_p^{5/3} R^5} \frac{R^3 \hbar^2}{m_e} = \underbrace{m_p^{-5/3} m_e^{-1} \hbar^2}_{\text{constants}} M_*^{5/3} \frac{1}{R^2}$$

fixed for fixed M

$$E_{star} \sim E_{kin} + E_{grav}$$

$$= -\frac{GM^2}{R} + \frac{\hbar^2 m_e^{-1} m_p^{-5/3} M^{5/3}}{R^2}$$

$$E = -\frac{\alpha}{R} - \frac{\beta}{R^2} \qquad \alpha : GM^2$$

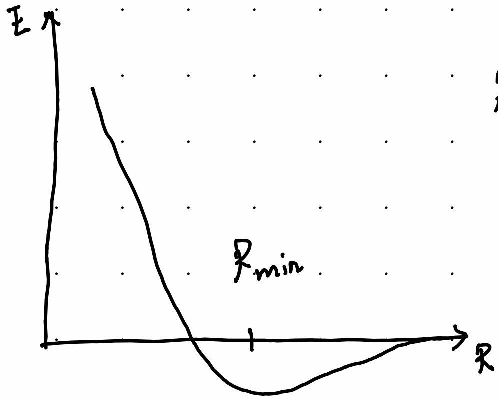
$$\beta := -\hbar^2 m_e^{-1} m_p^{-5/3} M^{5/3}$$

At equilibrium star is at minimum. So what is the smallest possible radius for a star in equilibrium?
↳ supported by electron degeneracy?

$$\frac{dE}{dR} = 0 \Rightarrow +\frac{\alpha}{R^2} + 2\frac{\beta}{R^3} = 0$$

$$\Rightarrow -\frac{\alpha}{R^2} = \frac{2\beta}{R^3} \Rightarrow \boxed{R = -\frac{2\beta}{\alpha}}$$

$$R = \frac{2\hbar^2 m_e^{-1} m_p^{-5/3} M^{5/3}}{GM^2} = \frac{2\hbar^2 m_e^{-1} m_p^{-5/3}}{G} M^{-1/3}$$




$$R_{min} = \frac{\hbar^2 M^{-1/3}}{G\, m_e\, m_p^{5/3}}$$

This means that energy is bounded from below and the star does not collapse.

But what happens if we let M vary, become more massive? The electrons beome relativistic.

$$\langle E \rangle = \langle p_e \rangle c = \hbar n_e^{1/3} c$$

$$E_{kin} \sim n_e R^3 \langle E \rangle \sim \hbar n_e^{4/3} R^3 c$$

$$\sim \frac{\hbar R^3 M^{4/3}}{m_p^{4/3} R^4} \sim \frac{\hbar m_p^{-4/3} M^{4/3}}{R}$$

So: $E = -\frac{\alpha}{R} + \frac{\gamma}{R}$ $\Rightarrow$ Equilibrium is possible only for $\alpha = \gamma$.

$$\Rightarrow \quad \frac{GM^2}{\cancel{R}} = \hbar c \, \frac{M^{4/3}}{m_p^{4/3}} \, \frac{1}{\cancel{R}}$$

$$\Rightarrow \quad M^{2/3} = \hbar c \, m_p^{-4/3} \, G^{-1}$$

$$\boxed{M = \hbar^{3/2} c^{3/2} G^{-3/2} m_p^{-2}}$$

if we increase $M$ any further, $R$ will decrease an the star can no longer be supported by electron degeneracy.

Chandrasekhar mass $\sim 1.4 \, M_\odot$

## Neutron stars :

If electron degeneracy does not suffice, the next particle species to become degenerate is the proton. But at this point, protons are not "stable" in the presence of electrons.

This is due to the inverse beta decay :

$$e^- + p^+ \longrightarrow n + \nu_e$$

Why is this?

$\Delta m \sim m_n - m_p$.

$E_{inv-\beta} = \Delta m \, c^2$.

But $\Delta m > m_e$ $(\Delta m \sim 3 m_e)$.

So if we continue to crush the star, then there is a point where the Fermi energy of the particle species $\sim \Delta m \, c^2$. At this point, inverse $\beta$-decay is inevitable.

However, $n + \nu_e \longrightarrow e^- + p^+$ cannot dominate.

these guys escape at $\sim c$.

+ they swith flavour.

So the next state for the star is to be supported by neutron degeneracy.

The analysis follows as before (Homework).

$$R_c^{\text{electron}} = \frac{1}{\underbrace{m_e m_p}_{m_n^2}} \left(\frac{\hbar^3}{Gc}\right)^{1/2}$$

$$R_c^{\text{neutron}} \sim \frac{1}{m_n^2}\left(\frac{\hbar^3}{Gc}\right)^{1/2} \sim \frac{1}{m_p^2}\left(\frac{\hbar^3}{Gc}\right)^{1/2}$$

(Recall equilibrium
For $E = -\frac{\beta}{R} + \frac{\alpha}{R}$
$\sim m_p^2 = \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{M}$ )

$$\sim M\left(\frac{G}{\hbar c}\right)^{3/2} \frac{\hbar^{3/2}}{G^{1/2} c^{1/2}}$$

$$\sim \frac{GM}{c^2}$$

$\rightsquigarrow$ Schwarzschild radius.

So we need to do a careful GR calculation of a perfect fluid undergoing grav. collapse.

$$M_{max}^{neutron\text{-}deg} \sim 3 M_\odot$$

if $M > M_{max}^{neut\cdot deg} \longrightarrow$ ? $\longrightarrow$ Black hole.

Quark stars?

## Tolman-Oppenheimer-Volkoff bound:

The TOV equation tells you how pressure changes as a function of R, M, energy density, for a spherically symmetric object.

Consider a spherically symmetric system.

Metric (in spherical coordinates)

$$g_{\mu\nu} = \operatorname{diag}\left(e^{f(r)}, -e^{g(r)}, -r^2, -r^2 \sin^2\theta\right)$$

$f(r)$, $g(r)$ are as of yet unknown, time-independent isotropic functions.

Plug this metric into the Einstein-field eqs.

Homework:

$$ds^2 = e^{f(r)}\,dt^2 - e^{g(r)}\,dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

Compute $\Gamma^{\alpha}_{\mu\nu}$. Compute $R^{\mu}{}_{\nu\alpha\beta}$.

Compute $R^{\mu}{}_{\alpha\mu\beta}$ to get $R_{\alpha\beta}$. Compute $R$.

Compute $\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{G_{\mu\nu}} = T_{\mu\nu}$.

$$G_{00} = \frac{1}{r^2}\left[1 - \frac{d}{dr}\left(r\, e^{-g(r)}\right)\right] e^{f(r)} = T_{00}$$

But if we have a spherically symmetric gas ball: we know what its stress energy tensor looks like:

$$T_{\mu}{}^{\nu} = \begin{bmatrix} \overset{\text{energy}}{\epsilon(r)} & 0 & 0 & 0 \\ 0 & -P(r) & 0 & 0 \\ 0 & 0 & -P(r) & 0 \\ 0 & 0 & 0 & -P(r) \end{bmatrix} \rightarrow \text{pressure}$$

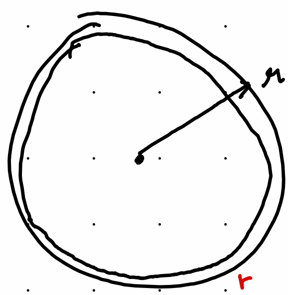
$$T_{\mu\nu} = T_\mu^{\ \alpha} g_{\alpha\nu}.$$

$$= \epsilon(r)\, g_{00} = \overset{8\pi G}{K}\, \epsilon(r)\, e^{f(r)}$$

$$\frac{1}{r^2}\left[1 - \frac{d}{dr}\left(r e^{-g(r)}\right)\right] e^{f(r)} = K\, \epsilon(r)\, e^{f(r)}.$$

$$= 1 - \frac{d}{dr}\left(r e^{-g(r)}\right) = K r^2 \epsilon(r)$$

Consider.




$$\frac{dM}{dr} = 4\pi\, \epsilon(r)\, r^2$$

$$\int_0^r 1 - \frac{d}{dr}\left(r e^{-g(r)}\right) dr = \int_0^r \frac{K}{4\pi} \frac{dM}{dr}\, dr$$

Classical GR

$$r - r e^{-g(r)} = \frac{K}{4\pi}\left(M(r) - \cancel{M(0)}^{\,0}\right)$$

$$\boxed{1 - \frac{K}{4\pi r} M(r) = e^{-g(r)}}$$

Now consider

$G_{11}$ and $T_{11}$

$$\frac{1}{r^2}\left(r\frac{d}{dr}f(r) - e^{g(r)} + 1\right) = K\,P(r)\,e^{g(r)}$$

$$f'(r) = r\,K\,P(r)\,e^{g(r)} - \frac{1}{r} + \frac{e^{g(r)}}{r}$$

$$= \frac{r\,K\,P(r)}{1 - \frac{K}{4\pi r}M(r)} - \frac{1}{r} + \frac{1}{\left(1 - \frac{K}{4\pi r}M(r)\right)r}$$

$$= \frac{r^2 K P(r) - \left(1 - \frac{K}{4\pi r}M(r)\right) + 1}{\left(1 - \frac{K}{4\pi r}M(r)\right)r}$$

$$= \frac{r^2 K P(r) \cancel{-1} + \frac{K}{4\pi r}M(r) \cancel{+1}}{\left(1 - \frac{K}{4\pi r}M(r)\right)r}$$

$$f'(r) = \frac{r^2 K P(r) \cancel{-1} + \frac{K}{4\pi r} M(r) \cancel{+1}}{\left(1 - \frac{K}{4\pi r} M(r)\right) r}$$

$$= \frac{\cancel{r} K \left( P(r) r + \frac{M(r)}{4\pi r^2} \right)}{\cancel{r} \left( 1 - \frac{K}{4\pi r} (M(r)) \right)}$$

$$f'(r) = K \left( P(r) r + \frac{M(r)}{4\pi r^2} \right) \left( 1 - \frac{K}{4\pi r} M(r) \right)^{-1} \quad \star$$

Also, $\nabla_\mu T^\mu{}_\nu = 0$ (energy-momentum conservation)

$$\nabla_\mu T^\mu{}_1 = 0 \Rightarrow f'(r) = - \frac{dP(r)}{dr} \frac{1}{P(r) + \varepsilon(r)}$$

(check)

Equate with $\star$,

$$\frac{dP}{dr} = -\left(P(r) + \varepsilon(r)\right) K \left( P(r) r + \frac{M(r)}{4\pi r^2} \right) \left( 1 - \frac{K}{4\pi r} M(r) \right)^{-1}$$

Put it altogether and re-introducing $G$.

$$\frac{dP}{dr} = -\frac{G\,\epsilon(r)\,m(r)}{c^2 r^2}\left(1+\frac{P(r)}{\epsilon(r)}\right)\left(1+\frac{4\pi r^3 P(r)}{M(r)\,c^2}\right)\left(1-\frac{2GM(r)}{c^2 r}\right)^{-1}.$$

not exactly Schwarzschild as we saw earlier.

$\downarrow$

Tolman-Oppenheimer-Volkov equation.

Maximum mass for neutron stars; integrating the TOV equation and working some equation of state of a neutron star.

$M_{TOV} \sim (1.5-3)\,M_\odot$. But this is difficult to pin down since equations of state of neutron stars are difficult to derive, lots of unknowns. (phase of matter....). Hope is that GW astronomy will be of use here. But that's all we will discuss here for now.

# The Schwarzschild Solution:

(vacuum)

Static, spherically symmetric solutions to Einstein's equations

Spherically symmetric: Isometry group contains $SO(3)$.

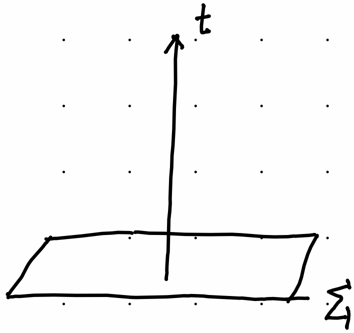
time independent: A spacetime manifold $\mathcal{M}$ is stationary if it admits a killing vector field $K_\mu$ such that $g_{\mu\nu} K^\mu K^\nu < 0$.

Def: A killing vector field $X$ is a vector field such that $\mathcal{L}_X g = 0$, $\mathcal{L}$ = Lie derivative, $g$: metric.

$$(\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0)$$

How to choose coordinates on $\mathcal{M}$?

Consider a hypersurface $\Sigma_1$ such that $K^a$ is nowhere tangent to this hypersurface.




Introduce coordinates $x^i$ on $\Sigma_1$.

$K^a = \left(\frac{\partial}{\partial t}\right)^a$. Since $K^a$ is a Killing vector field, the metric is independent of $t$.

$$ds^2 = g_{00}\, dt^2 + 2 g_{0i}\, dt\, dx + g_{ij}\, dx^i\, dx^j$$

Hypersurface orthogonality:

Let $\Sigma_1$ be a hypersurface in $M$ specified by $f: M \to \mathbb{R}$, $f(x) = 0$, $df \neq 0$ on $\Sigma_1$.

Then the 1-form $df$ is normal to $\Sigma_1$.

<u>Proof:</u> Let $t^a$ be a tangent vector to $\Sigma_1$. Then $df(t) = t(f) = t^\mu \partial_\mu f = 0$ (because $f$ is constant on $\Sigma_1$). Another one form $n$ normal to $\Sigma_1$ can be written as $n = g\, df + f n'$,

$g$ = smooth function, $n'$ is a smooth 1-form.

$dn = dg \wedge df + df \wedge n'$. So $(dn)_{\Sigma_1} = (dg - n') \wedge df$.

if $n$ is normal to $\Sigma_1$ then $(n \wedge dn)_{\Sigma_1} = 0$.

Converse :

Theorem (Frobenius) :

if $n$ is a non-zero 1-form such that $n \wedge dn = 0$ everywhere, then $\exists$ functions $f, g$ st $n = g\,df$. Such that $n$ is normal to surfaces of constant $f$. We then say that $n$ is hypersurface-orthogonal.

Definition : A spacetime $M$ is static if it admits a hypersurface-orthogonal timelike Killing field.

Let's go back to

$$ds^2 = g_{00}\,dt^2 + 2g_{0i}\,dx^i dt + g_{ij}\,dx^i dx^j$$

at $t=0$; $\Sigma_1$ is the chart $x_i$; $K_\mu \propto (1, 0, 0, 0)$.

This means $g_{0i}$ must be zero.

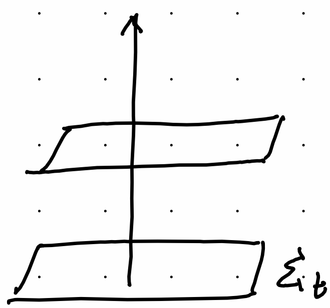
$$ds^2 = g_{00}\, dt^2 + g_{ij}\, dx^i dx^j \,, \quad (g_{00} < 0)$$

Static $\Rightarrow$ stationary + invariant under $t \to -t$.

Rotating spacetimes are not static since the direction of rotation changes. To derive the metric of a static, spherically symmetric object:

Isometry group is $\mathbb{R} \times SO(3)$.

Birkhoff's theorem: if $M$ has isometry $\mathbb{R} \times SO(3)$, then it is static.




$$ds^2\Big|_{t = \text{const}} = e^{2\psi(r)} dr^2 + r^2 d\Omega_2^2$$

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$\downarrow$

metric under orbits of $SO(3)$.

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Psi(r)} dr^2 + r^2 d\Omega_2^2$$

Consider inside the star an ideal fluid:
The behaviour of the fluid is something we have already encountered. (TOV equations).

$$g_{\mu\nu} = \mathrm{diag}\left(-e^{2\Phi(r)}, e^{2\Psi(r)}, r^2, r^2\sin^2\theta\right)$$

HW: Compute the Einstein equations
(tt, rr, θθ)

$$T_{tt} \Rightarrow \frac{dM}{dr} = 4\pi r^2 \rho$$

$$T_{rr} \Rightarrow \frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{(r-2m)\,r}$$

$$T_{\theta\theta} \Rightarrow -(p+\rho)\,\frac{(m + 4\pi r^3 p)}{r(r-2m)}$$

(These are the TOV equations again).

$$T_{rr} \Rightarrow \frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{(r-2m)r}$$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{(r-2m)r}$$

$$\int d\Phi = \int dr \frac{M}{(r-2m)r} + \int dr \frac{4\pi r^2 P}{(r-2m)}$$

Assuming constant shell density,

$$M(r) \to M. \quad \lim_{P\to 0} \Phi = \frac{1}{2}\log\left(\frac{-2M+r}{r}\right)$$

$r > R,\ P \to 0$

$$= \frac{1}{2}\log\left(1 - \frac{2M}{r}\right)$$

$$\boxed{-e^{2\Phi(r)} = g_{tt} = -\left(1 - \frac{2M}{r}\right)}$$

Since the metric must asymptotically scale to minkowski, $e^{2\Phi(r)} \cdot e^{2\Psi(r)} = 1$.

$$e^{\psi(r)} = \frac{1}{\left(1 - \frac{2M}{r}\right)}$$

$$\boxed{ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2}$$

This metric describes the metric outside any spherically symmetric, static object.
if you evaluate $T_{\mu\nu}$ for this metric, you will see that it is zero. "Vacuum solutions".

So, when) why does this become problematic?
Given a mass $M$,

$\frac{2GM}{c^2} = R_s$. So for $r > R_s$ $ds^2_{schw}$ is well defined.

# II: The Schwarzschild spacetime

The $ds^2_{sch}$ is also the metric around a star following collapse into a static, spherically symmetric ball.

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

On the surface, if we let $r = R(t)$.

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{\partial R}{\partial t}dt\right)^2 + r^2 d\Omega_2^2$$

$$= \left[-\left(1-\frac{2M}{R}\right) + \left(1-\frac{2M}{R}\right)^{-1}\dot{R}^2\right]dt^2 + R^2 d\Omega_2^2$$

On radial geodesics, $d\theta = d\phi = 0$

$\therefore d\Omega^2 = 0$

$$ds^2_{rad} = \left(-\left(1-\frac{2M}{R}\right) + \left(1-\frac{2M}{R}\right)^{-1}\dot{R}^2\right)dt^2$$

$$ds^2 = -d\tau^2$$

$$1 = \left[\left(1-\frac{2M}{R}\right) - \left(1-\frac{2M}{R}\right)^{-1}\dot{R}^2\right]\left(\frac{dt}{d\tau}\right)^2$$

But since $\partial_t$ is a Killing vector, we have conservation of energy

$$\epsilon = -g_{00}\frac{dt}{d\tau} \Rightarrow \text{conserved}$$

Plug in

$$\left(1-\frac{2M}{R}\right)\frac{dt}{d\tau} = \epsilon$$

$$1 = \left[\left(1-\frac{2M}{R}\right) - \left(1-\frac{2M}{R}\right)^{-1}\dot{R}^2\right]\epsilon^2\left(1-\frac{2M}{R}\right)^{-2}$$

$$\left(1-\frac{2M}{R}\right) - \left(1-\frac{2M}{R}\right)^2\frac{1}{\epsilon^2} = \left(1-\frac{2M}{R}\right)^{-1}\dot{R}^2$$

$$\left(1-\frac{2M}{R}\right)^2 - \left(1-\frac{2M}{R}\right)^3\frac{1}{\epsilon} = \dot{R}^2$$

$$\epsilon^2\left(1-\frac{2M}{R}\right)^2\left(\epsilon^2 - 1 + \frac{2M}{R}\right) = \dot{R}^2$$

$$\underbrace{\epsilon^2\left(1-\frac{2M}{R}\right)^2}_{\text{decreases faster}}\underbrace{\left(\epsilon^2-1+\frac{2M}{R}\right)}_{>0\text{, decreasing}} = \dot{R}^2$$

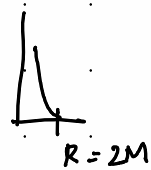
$e < 1.$

$\epsilon^2 \ll 1$

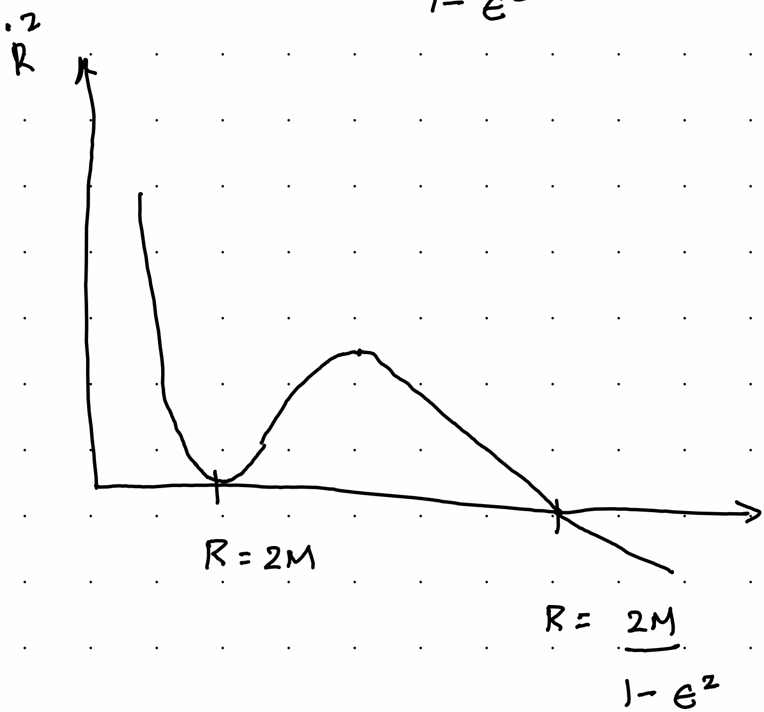
$\frac{2M}{R} > 1$

For $R < 2M$,

$$\dot{R}^2 \simeq \left(1-\frac{2M}{R}\right)^2$$




For $R > 2M$ ; $R < \frac{2M}{1-\epsilon^2}$

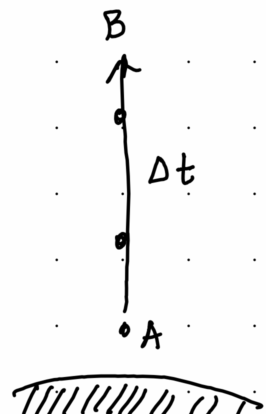



So what does this mean?

An observer seeing the star collapse notes that the collapse starts off at $R_{max} = \frac{2M}{1-\epsilon^2}$ at zero velocity. Then collapse to $R = 2M$ But no further. This can also be seen in terms of gravitational redshift.

Consider two observers A and B.

$(r_A, \theta, \phi)$ , $(r_B, \theta, \phi)$. Let $r_B > r_A$.




A sends 2 photons to B $\Delta t$ apart. These photons travel along the same path. Why?

$\partial_t$ is a Killing vector.

B, $r_B$.

For: $d\tau^2 = \left(1 - \frac{2M}{R}\right) dt^2$

$$\Delta\tau_A = \left(1 - \frac{2M}{r_A}\right)^{1/2} \Delta t$$

Similarly, look @ how long b/w B receives the 2 photons.

$$\Delta\tau_B = \left(1-\frac{2M}{r_B}\right)^{1/2}\Delta t$$

$$\left[\frac{(r_B-2M)r_A}{(r_A-2M)r_B}\right]^{1/2}$$

$2Mr_B > 2Mr_A$.

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \left(\frac{1-2M/r_B}{1-2M/r_A}\right)^{1/2} > 1.$$

Now instead of photons, what if A sends light waves?

The above formula can be applied to the time b/w 2 successive wave crests. In $c=1$ units, $\Delta\tau = \lambda$

$\Rightarrow \lambda_B > \lambda_A$ [Redshift]

if $B \gg 2M$,

$$1+z = \frac{\lambda_B}{\lambda_A} \simeq \frac{1}{\left(1-\frac{2M}{r_A}\right)^{1/2}}.$$

$\lim_{r_A \to 2M} \Rightarrow \infty$ red shift..

So things appear to approach the $r=2M$ hypersurface but eventually just fade away.

r = 2M is a "special" place. Event horizon.

But what happens to an observer who actually falls in?

The propertime variable is along the radial geodesic.

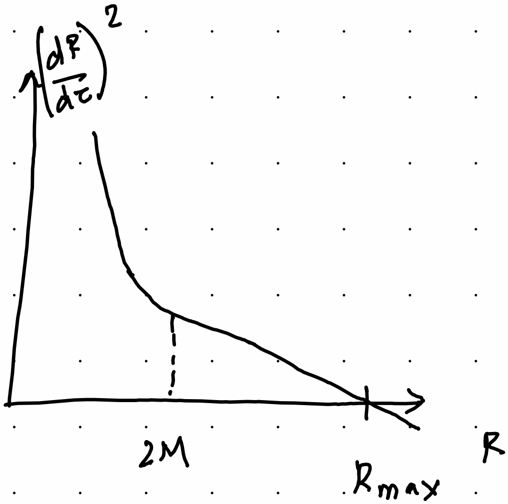
$$\frac{d}{dt} = \left(\frac{dt}{d\tau}\right)^{-1} \frac{d}{d\tau} = \frac{1}{\epsilon}\left(1 - \frac{2M}{R}\right)\frac{d}{d\tau}$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{1}{\epsilon^2}\left(1 - \frac{2M}{R}\right)^2\left(\frac{2M}{R} - 1 + \epsilon^2\right)$$

$$\left(\frac{1}{\epsilon}\left(1 - \frac{2M}{R}\right)\frac{dR}{d\tau}\right)^2 = \frac{1}{\epsilon^2}\left(1 - \frac{2M}{R}\right)^2\left(\frac{2M}{R} - 1 + \epsilon^2\right)$$

$$\cancel{\frac{1}{\epsilon^2}\left(1 - \frac{2M}{R}\right)^2}\left(\frac{dR}{d\tau}\right)^2 = \cancel{\frac{1}{\epsilon^2}\left(1 - \frac{2M}{R}\right)^2}\left(\frac{2M}{R} - 1 + \epsilon^2\right)$$

$$\left(\frac{dR}{d\tau}\right)^2 = (1-\epsilon^2)\left(\frac{R_{max}}{R} - 1\right), \quad R_{max} = \frac{2M}{1-\epsilon^2}$$




The observer in his/her/their frame falls through $R = 2M$ in finite propertime. Nothing happens at $R = 2M$.

<u>Homework</u>: Consider a supermassive black hole $M \sim 10^9 M_\odot$. Assume it is not accreting/active. You decide to fall in (because of high cost of living). How long in propertime before you reach the singularity?

Consider an infalling light ray.
radial, null.

$$ds^2 = 0 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

$$dt^2 = \frac{1}{\left(1 - \frac{2M}{r}\right)^2} dr^2 = dr^{*2}$$

where $r^* = r + 2M \ln \left| \frac{r - 2M}{2M} \right|$.

"Regge-Wheeler coordinate / tortoise coordinate"

Ingoing Eddington-Finkelstein.

$r = [2M, \infty)$, $r^* = (-\infty, \infty)$ in the Regge-Wheeler coordinate.

$d(t \pm r^*) = 0$ on radial null geodesics.

We define the ingoing radial null coordinate as $v = t + r^*$, $v \in (-\infty, \infty)$

in the new "ingoing EF coordinates", the Schwarzschild metric becomes:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

Along radial null geodesics,

$$\frac{dt}{dr} = \frac{1}{\left(1 - \frac{2M}{r}\right)}, \quad \frac{dr}{\left(1 - \frac{2M}{r}\right)} = dr^*$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

Since $v = t + r_*$ $\quad dt = dv - \dfrac{dr}{\left(1 - \frac{2M}{r}\right)}$

$$dt^2 = dv^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} - \frac{2\,dv\,dr}{\left(1 - \frac{2M}{r}\right)}$$

$$-\left(1 - \frac{2M}{r}\right)dt^2 = -\left(1 - \frac{2M}{r}\right)dv^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + 2\,dv\,dr$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega_2^{\,2}$$

$(v, r, \theta, \phi) \Rightarrow$ Eddington - Finkelstein coordinates

These coordinates are smooth @ $r = 2M$.

$$g_{\mu\nu}^{EF} = \begin{pmatrix} -(1-2M/r) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The metric is smooth $\forall\, r > 0$.

Non-degenerate for $r > 0$, Lorentzian for $r > 0$.

The black hole solution can be extended until $r = 0$, through the EH.

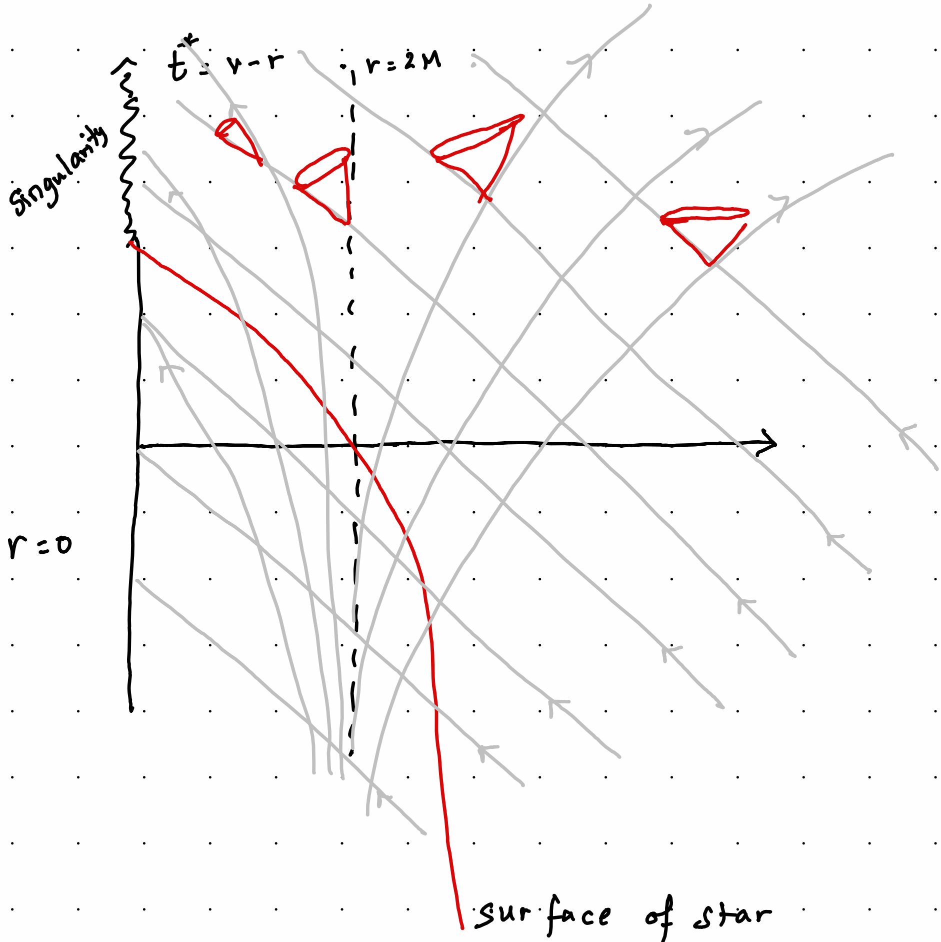
$r = 2M$ in Schwarzschild $\rightarrow$ "Coordinate singularity"

Not enough charts to cover a manifold.

$r = 0 \Rightarrow$ Curvature singularity [Geodesic incompleteness]

$$R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = \frac{48M^2}{r^6}$$ (Kretschmann Scalar)

Finkelstein diagram:




All light cones point inwards and towards the singularity, once the singularity has been formed.

(This is why nothing escapes a black hole).

When $r < 2M$,

$$2\,dr\,dv = \left[\cancelto{0}{ds^2} + \left(1 - \frac{2M}{r}\right) dv^2 + \cancelto{0}{r^2 d\Omega_2^2}\right]$$

radial, null:

$$2\,dr\,dv = \underbrace{\left(1 - \frac{2M}{r}\right)}_{<0 \text{ if } r < 2M.} dv^2$$

$\Rightarrow$ $-\left(1 - \frac{2M}{r}\right) dv^2$, which is supposed to behave timelike, behaves space like.

So, an infalling observer <u>WILL</u> hit the singularity in finite proper time.

# The outgoing Eddington-Finkelstein coordinate

It seems unusual that the Event horizon is a region from which no null / timelike geodesics can escape. Einstein field equations are time reversible. Let's see how time reversal works:

We define the outgoing radial null coordinate:

$u = t - r_*, \quad -\infty < u < \infty$

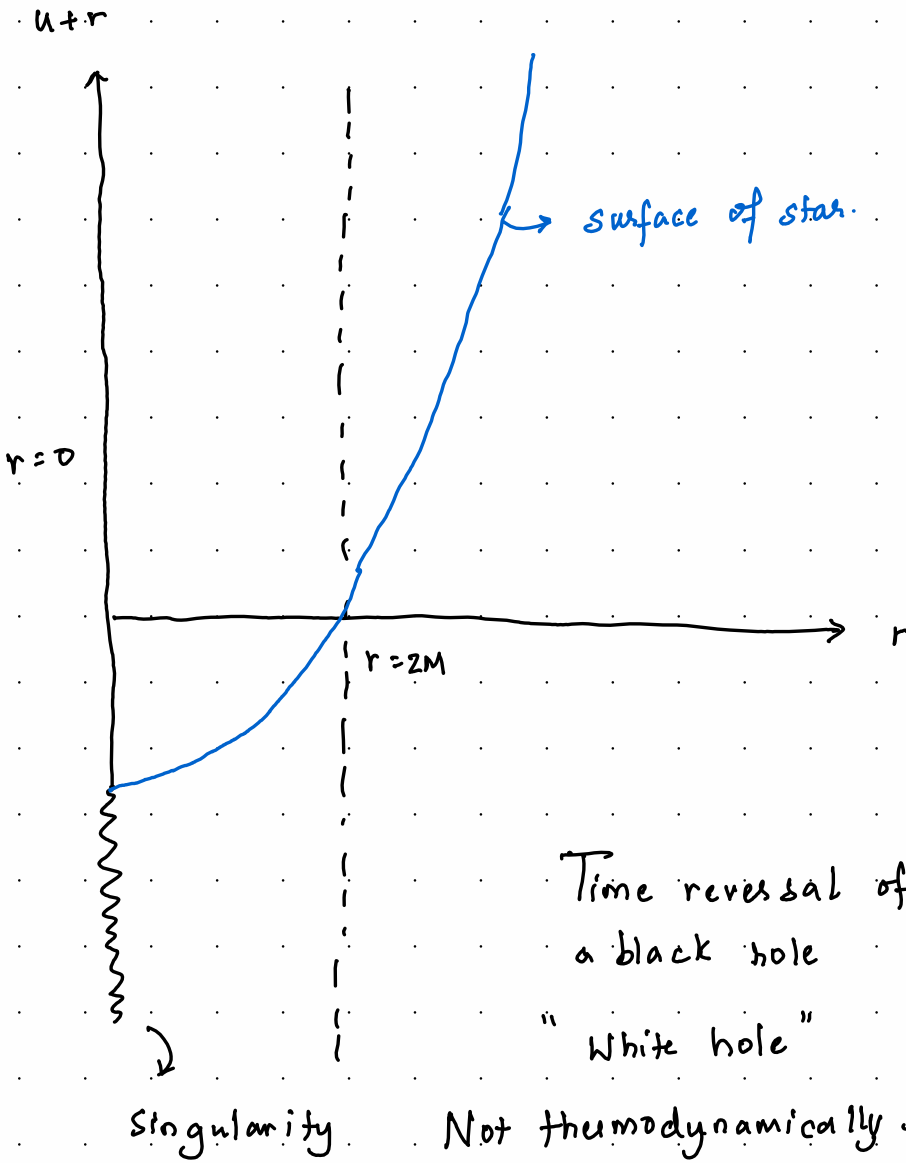
$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2\,du\,dr + r^2 d\Omega_2^2$$

In these coords, the $r < 2M$ (OEFC) is not the same as $r < 2M$ (IEFC).

The difference (check) is due to

$2\,du\,dr \geq 0$ on time like / null geodesics.

A star with surface $r < 2M$, expands through $r = 2M$.




Time reversal of a black hole

"White hole"

Not thermodynamically stable.

Defining a black hole:

<u>Def</u>: a vector $v$ is causal if it is timelike or null.

ie. $g_{\mu\nu} v^\mu v^\nu \leq 0$.

$g_{\mu\nu}$ has signature $(-,++\dots+)$

A curve $C$ is causal if $\forall\ p \in C$, the tangent vector $\dot{v}_\mu\big|_p$ is causal

<u>Def</u>: If a spacetime $M$ admits a causal vector field $T^a$, it is time-oriented. if $X^a$ is another vector field and $X^a$ lies in the light cone of $T^a$, $X^a$ is future-oriented. Else it is past oriented.

<u>Note</u>: For $r > 2M$, we may choose $K = \frac{\partial}{\partial t}$ to be our time-orientation. But it is not valid for $r < 2M$. since $K = \frac{\partial}{\partial v}$ is spacelike. ⚠

Instead: choose $\frac{\partial}{\partial r}$ for $r > 0$.

Consider a spacetime manifold $\mathcal{M}$. A submanifold $B \subset \mathcal{M}$ such that no null vectors in $B$ can be continued to the rest of $\mathcal{M}$ is a black hole.

$\partial B$ is the event horizon.

Often there are other horizons that may play a role in GR / BH physics (trapping horizons, apparent horizons but we do not discuss them here. In fact the EH is much more formal and involves enegy conditions ...)

<u>Claim</u>: $\nexists$ no future directed causal cure that connects a point in $r \leq 2M$ to a point $r > 2M$.

<u>Proof</u>: Let $x^\mu(\lambda)$ be a future directed causal curve. Let $r(\lambda_0) \leq 2M$. We need to show that $\forall\ \lambda > \lambda_0,\ r(\lambda) \leq 2M$.

<u>Homework</u>: Or come ask me later for a proof.

# Kruskal–Szekeres coordinates

IEFC and OEFC both cover $r > 2M$.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv + r^2 d\Omega^2$$

$v = t + r_*$

$u = t - r_*$

Introduce $U, V$ for $r > 2M$

$u - v = -2r_*$

$= r + 2M \ln \left(\frac{r-2M}{2M}\right)$

$$U = -e^{-u/4M}$$

$$V = e^{v/4M}$$

$$dU = \frac{e^{-u/4M}}{4M} du, \quad dV = \frac{1}{4M} e^{v/4M} dv$$

$$4M e^{u/4M} dU = du, \quad 4M e^{-v/4M} dV = dv$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) 16M^2 e^{(u-v)/4M} dU\, dV + r^2 d\Omega^2$$

$\downarrow$

$$ds^2 = -\frac{32M^3}{r(u,v)} e^{-\frac{r(u,v)}{2M}} dU\, dV + r(u,v)^2 d\Omega_2^2$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) 16M^2 e^{(u-v)/4M} \, dU \, dV + r^2 d\Omega^2$$

$$e^{(u-v)/4m} = e^{-2r_*/4M} = e^{-\frac{2}{4M}\left(r + 2M \ln\left(\frac{r-2M}{2M}\right)\right)}$$

$$e^{-r/2M} \, e^{-\frac{\cancel{4M}}{\cancel{4M}}\left(\ln\left(\frac{r-2M}{2M}\right)\right)}$$

$$= e^{-r/2M} \, e^{-\ln\left(\frac{r-2M}{2M}\right)}$$

$$= e^{-r/2M} \, e^{\ln\left(\frac{2M}{r-2M}\right)} = e^{-r/2M}\left(\frac{2M}{r-2M}\right)^{1/2}$$

$$ds^2 = -\left(\frac{\cancel{r-2M}}{r}\right) 16M^2 \left(\frac{2M}{\cancel{r-2M}}\right) e^{-r/2M} \, du \, dV + r^2 d\Omega_2^2$$

$$ds^2 = -\frac{32M^3}{r(u,v)} e^{-r(u,v)} \, du \, dV + r^2 d\Omega_2^2$$

Metric of a Schwarzschild BH
in Kruskal-Szekeres coords.

The metric is initially defined for $u<0, v>0$ but can be extended to $u>0, v<0$.

Horizon $r=2M \longrightarrow UV = e^{(u-v)/4M}$

$\downarrow$

$uv = 0$

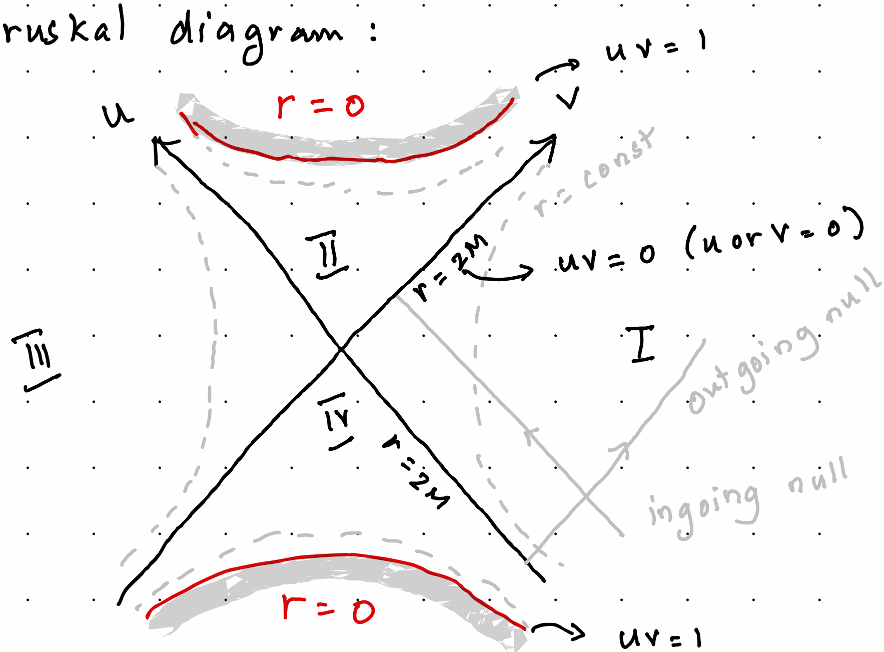
$$= e^{-\frac{2}{4m}(2M + 2M\ln(0))}$$

$$= e^{-1} e^{\ln(0)} = 0$$

Singularity $r=0 \longrightarrow uv = -e^{-r/2M}\left(\frac{r}{2M} - 1\right)$

$$= -(-1) = +1$$

Kruskal diagram:



In the KS coordinates, the EH and singularities are two surfaces. "Bifurcate"

↳ we will come back to this later.

Region I : $r > 2M$

Region II : The Schwarzschild black hole

Region III : A completely new region that is isometric to I. $(u,v) \rightarrow (-u,-v)$

But for I and III to "talk" to each other, geodesics need to go through the singularity, which is impossible.

Region IV : White hole (outgoing EF coordinates)

## Singularities, ER bridges and Eternal BH's

Consider $\dfrac{V}{u} = -e^{t/2M}$. Since it is again a monotonic function, it determines $t$ uniquely.

A line of constant $t$ is a straight line through the origin in the Kruskal diagaram.

These lines of constant $t$ therefore extend into $\overline{\underline{III}}$.

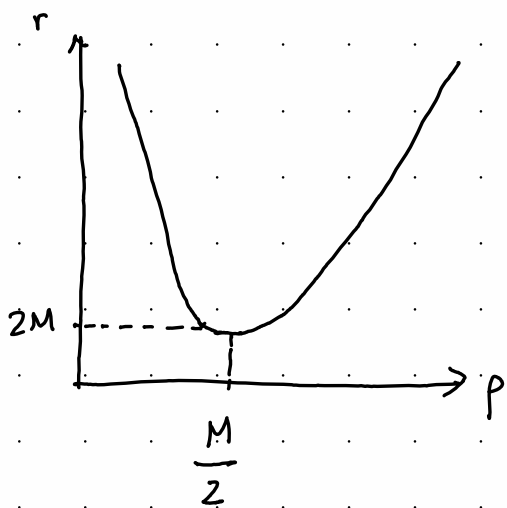
We want to investigate these constant $t$ hypersurfaces.

Let us define a new coordinate $\rho$ as:

$r = \rho + M + \dfrac{M^2}{4\rho}$. For a fixed $r$, we have 2 solutions for $\rho$.

$$\frac{dr}{d\rho} = 0 \Rightarrow 1 - \frac{M^2}{4\rho^2} = 0 \Rightarrow 4\rho^2 = M^2$$

$$\rho = \xrightarrow{\text{choose}} + \frac{M}{2}.$$




$$r\left(\rho = \frac{M}{2}\right) = \frac{M}{2} + M + \frac{M^{\cancel{2}}}{\cancel{4}\cancel{M}\,2} = \frac{M}{2} + M + \frac{M}{2} = 2M$$

"Eternal BH's"

all 4 regions are valid.

Stellar collapse not $t$ reversal symm.

$$ds^2 = \underbrace{-\left(1 - \frac{2M}{r}\right)}_{-\frac{(1 - M/2\rho)^2}{(1 + M/2\rho)^2}} dt^2 + \frac{dr^2}{\underbrace{\left(1 - \frac{2M}{r}\right)}} + \underbrace{r^2 d\Omega^2}$$

$$-\frac{\left(1 - M/2\rho\right)^2}{\left(1 + M/2\rho\right)^2} dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 d\rho^2 + \rho^2 \left(1 + \frac{M}{2\rho}\right)^4 d\Omega_2^2$$

$$ds^2 = -\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2} dt^2 + \left(1+\frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega_2^2\right)$$

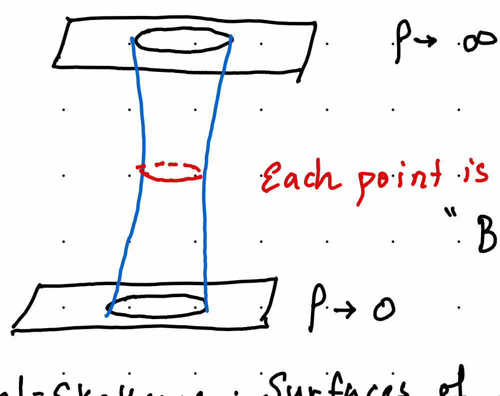
$$\rho \to \frac{M^2}{4\rho} \Rightarrow \underline{\overline{\mathrm{I}}} \to \underline{\overline{\mathrm{III}}}.$$

Metric has a singularity @ $\rho = M/2$. (coordinate)

Constant $t$ surfaces:

$$ds^2 = \left(1+\frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega_2^2\right)$$

Non-singular for $\rho > 0$. It describes a 3-manifold with topology $\mathbb{R} \times S^2 \hookrightarrow M_{3,1}$. For the time being let's "supress $\theta$".




" But this becomes singular, -ve $\varepsilon$-density required".

Kruskal-Skekues : Surfaces of constant $t$ are ER bridges | wormholes.

Kruskal-Szekeres is "global" / maximal

A spacetime $(\mathcal{M}, g)$ is extendible if $\exists\, (\mathcal{M}', g')$ whose submanifold $(\mathcal{U}, g|_{\mathcal{U}})$ is isometric to $(\mathcal{M}, g)$. Schwarzschild $\longrightarrow$ EF $\longrightarrow$ Kruskal.

Kruskal is inextendible. (maximal analytic extension).

## A note on singularities:

- Curvature (Big bang, BH singularity)
- Coordinate (Schwarzschild event horizon)
- Conical

Diagnosis of a curvature singularity:

$R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$ $\rightarrow$ diverges (singular) / regular

$\downarrow$

Kretschmann scalar

$$K_{sch} = \frac{48M^2}{r^6}$$

regular @ $r = 2M$, div. @ $r = 0$.

$\downarrow$

Check this.

But look @ a cone. There is a singularity @ the tip but all curvature "invariants" are finite.

Geodesic incompleteness: A geodesic is complete if its affine parameter extends from $-\infty$ to $+\infty$. i.e the geodesic is well defined for all values of the affine parameter.

A spacetime on which all geodesics can be completed is a "geodesically complete spacetime".

Black hole spacetimes are not geodesically complete. Therefore a characterization of a BH singularity is that geodesics cannot be completed at the curvature singularity/ conical singularity.

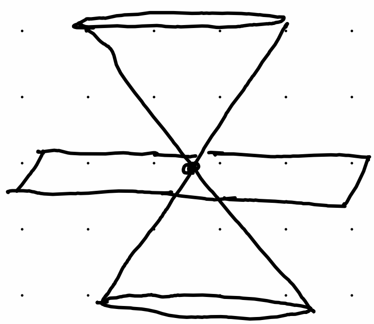
# III : Initial value problem and Singularity theorems

Are black holes inevitable? What asymptotic boundary conditions give rise to BH's? Can there be spacetimes without black holes?

Let $(M, g)$ be a time-oriented spacetime.
Recall that time oriented spacetime is a spacetime that admits a causal vector field. (i.e a timelike/null vector field)

A partial Cauchy surface $\Sigma_1$ is a hypersurface is a hypersurface for which no two points are causally connected.

Ex:



→ partial Cauchy H.S.

$\forall p \in \Sigma_{1\,\text{cauchy}}$, $\nexists$ $p' \in \Sigma_{1\,\text{cauchy}}$ s.t. $p' \in$ Positive LC$(p)$.

Def: The chronological / temporal future of $\Sigma_1$, $I^+(\Sigma_1)$, is the set of all points that can be reached from $\Sigma_1$ by future directed timelike curves.

$\forall\, p \in \Sigma_1$, $L^+(p)$ is the future) positive light cone of $p$. $I^+(\Sigma_1) = \bigsqcup_p L^+(p)$.

Def: The future domain of dependence of $\Sigma_1$, $D^+(\Sigma_1)$, is the set $p \in M$ s.t. every past-directed inextendible (no endpoints) curve intersects $\Sigma_1$.

(it is the set of future events determined solely from the data on $\Sigma_1$.

Q: is $D^+(\Sigma_1) \subset I^+(\Sigma_1)$ or $I^+(\Sigma_1) \subset D^+(\Sigma_1)$ or $D^+(\Sigma_1) = I^+(\Sigma_1)$. What about for de Sitter space (expanding universe)?

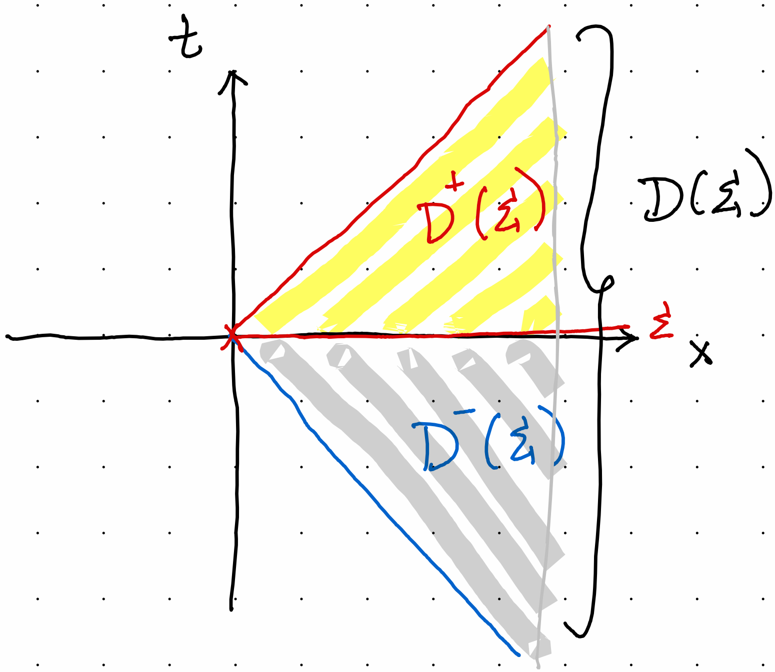
$D^+(\Sigma_1)$ is a shrinking set, while $I^+(\Sigma_1)$ is an increasing set.

**Def**: The future domain of dependence of $\Sigma_1$, $D^-(\Sigma_1)$, is the set of all $p \in M$ st every future extendible causal curve intersects $\Sigma_1$.

**Def**: Domain of dependence, $D(\Sigma_1)$,

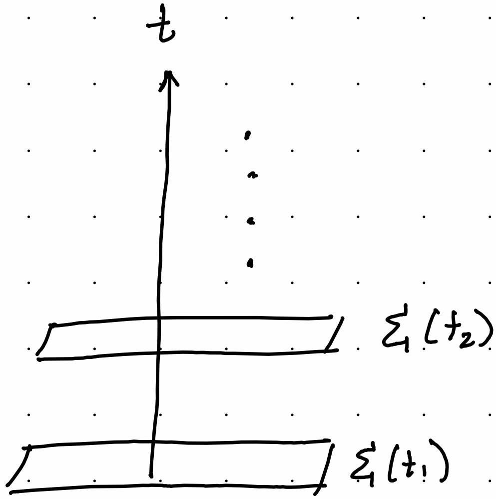
$$D(\Sigma_1) = D^+(\Sigma_1) \cup D^-(\Sigma_1).$$

Ex 2d Minkowski



<u>Def</u>: A spacetime $(M, g)$ is globally hyperbolic if it admits a Cauchy surface ie if $\exists$ a partial Cauchy surface $\Sigma_i$ such that $M = D(\Sigma_i)$.

Basically, a globally hyperbolic spacetime is one in which every event can be predicted for the Cauchy surface $\Sigma_1$ at $-\infty$.




if $\forall\, t_i$, $\Sigma_1(t_i)$ is Cauchy, $M$ is a globally hyperbolic spacetime.

Ex: The surface of constant $t$ in Kruskal spacetime (Einstein-Rosen bridge) is a global time function and the Kruskal spacetime (regions I, II, III, IV) is globally hyperbolic.

Initial conditions: Defined on a spacelike hypersurface (timelike killing vectors are normal to it).

But what are these initial conditions?

Since $\Sigma_1$ is a codim 1 surface, we need to specify the Einstein equations on $\Sigma_1$.

Intrinsic geometry of $\Sigma_1$: pull back of $g$ on $\Sigma_1$.

$$\underbrace{ds^2}_{\text{ADM decomposition of } g \text{ (possible only for globally hyperbolic spacetimes)}} = -N^2 dt^2 + \underbrace{h_{ij}}_{\text{metric on } \Sigma_1} (dx^i + N^i dt)(dx^j + N^j dt)$$

How do you take time derivatives on $\Sigma_1$?

"Extrinsic curvature".

Let $n_a$ be the 1-form normal to $\Sigma_1$.

$$n^a n_a = -1.$$

Now, $h^a{}_b = \delta^a_b + n^a n_b$

if $X^a, Y^b$ are tangent to $\Sigma_1$ : $h_{ab} X^a X^b = g_{ab} X^a X^b$

$\boxed{h^a{}_b n^b = 0} \rightarrow$ check, $h^a{}_b = h^a{}_c h^c{}_b$

Decomposition of a spacetime vector:

$$X^a = \delta^a{}_b X^b = \underbrace{h^a{}_b X^b}_{\substack{X^a_{\parallel} \\ \text{tangent} \\ \text{to } \Sigma_1}} - \underbrace{n^a n_b X^b}_{\substack{X^a_{\perp} \\ \text{normal to } \Sigma_1}}$$

Let $N_a$ be normal to $\Sigma_1$ at $p$. Parallel transport $N_a$ along a curve with tangent vector $X^a$.

$\downarrow$ on $\Sigma_1$

$$X^b \nabla_b N_a = 0$$

The question is: Does $N_a$ remain normal to $\Sigma_1$?

Let $Y^a$ be another tangent vector at $p$.

$$Y^a N_a = 0 \text{ at } p.$$

Consider $X^b \nabla_b (Y^a N_a) = X^b \cancel{Y^a \nabla_b N_a}^{\,0}$

$+ X^b N_a \nabla_b Y^a$

$\therefore$ if $X^b N_a \nabla_b Y^a = 0$, then $Y^a N_a$ vanishes and $\therefore$ $N_a$ remains normal to $\Sigma$. Converse also holds.

__Def__: The extrinsic curvature $K_{ab}$ at $p \in \Sigma$ is $K(X, Y) = -n_a (\nabla_{X_\parallel} Y_\parallel)^a$; $X, Y$ are vector fields on $M$.

$$K_{ab} = h^p_a h^q_b \nabla_p n_q$$

__Proof__: $K(X, Y) = -n_a (\nabla_{X_\parallel} Y_\parallel)^a$

$$= -n_a X_\parallel^c \nabla_c Y_\parallel^a$$

$$= X_\parallel^c Y_\parallel^a \nabla_c n_a$$

$$= h^c_b X^b h^a_d Y^d \nabla_c n_a$$

$$K_{ab} X^a X^b = X^a Y^b \boxed{h^c_a h^d_a \nabla_c n_d} = K_{ab}.$$

$\square$

$K_{ab} = K_{ba} \longrightarrow$ Prove this.

Hint: take $n_a = g(df)_a$ i.e a time-function.

and compute $\nabla_c n_d$.

Note: $K_{ab}$ is also just $\frac{1}{2} \mathcal{L}_n h_{ab}$ i.e

the Lie derivative of $h_{ab}$ along $n_a$.

This is why we say this the time derivative of a hypersurface.

## Gauss - Codazzi equations

Consider a tensor at $p \in \Sigma_t$. Consider its projection under $h^a{}_b$. The tensor is invariant under projection if

$$T^{a_1 \dots a_r}{}_{b_1 \dots b_s} = h^{a_1}{}_{c_1} h^{a_2}{}_{c_2} \dots h^{a_r}{}_{c_r} h^{d_1}{}_{b_1} \dots h^{d_s}{}_{b_s} *$$

$$* \, T^{c_1 \dots c_r}{}_{d_1 \dots d_s}$$

This allows you to define tensors on a submanifold containing $p$.

The covariant derivative $D$ on $\Sigma_1$ is the projection of the covariant derivative on $\Sigma_1$.

$$D_a T^{b_1 \dots b_r}{}_{c_1 \dots c_s} = h^d{}_a \, h^{b_1 \dots b_r}{}_{d_1 \dots d_r} \, h^{f_1 \dots f_s}{}_{c_1 \dots c_s} \nabla_d T^{d_1 \dots d_r}{}_{f_1 \dots f_s}$$

Analogous to $\nabla$, $D$ is the Levi-Civita connection associated to $h_{ab}$. ($D_a h_{bc} = 0$, $D$ is torsion free).

Riemann tensor:

$$\tilde{R}^a{}_{bcd} = h^a{}_e \, h^f{}_b \, h^g{}_c \, h^i{}_d \, R^e{}_{fgi} - 2 K_{[c}{}^a K_{d]b}$$

Ricci: $\tilde{R}_{bd} = \tilde{R}^a{}_{bad}$

$$\tilde{R}^a{}_{bcd} X^b = 2 D_c D_d X^a - 2 D_d D_c X^a$$

Ricci scalar:

$$\tilde{R} = R + 2 R_{ab} n^a n^b - (K_a{}^a)^2 + K^{ab} K_{ab}$$

Codazzi eqn: $D_a K_{bc} - D_b K_{ac} = h^d{}_a h^e{}_b h^f{}_c n^g R_{defg}$

$$D_a K_{bc} h^{ac} - D_b K_{ac} h^{ac} = h^{ac} h^d_a h^e_b h^f_c n^g R_{defg}$$

$$= D_a K^a{}_b - D_b K = h^c_b R_{cd} n^d$$

$\hookrightarrow$ Also Codazzi equation.

The Codazzi (aka Gauss-Codazzi) equation relates the induced metric $h$ to the extrinsic curvature of a submanifold of a (pseudo) Riemannian manifold of codim $\geq 1$.

## Defining the Einstein equation on $\Sigma_1$.

$$G_{ab} = 8\pi T_{ab} \qquad [G, c = 1] \quad \text{(normal-normal)}$$

$$G_{ab} n^a n^b = 8\pi T_{ab} n^a n^b$$

$$: R_{ab} n^{ab} + \frac{1}{2} R \underbrace{h_{ab} n^a n^b}_{1} =$$

$$\cancel{R_{ab} n^{ab}} + \frac{1}{2}(R' - \cancel{2 R_{ab} n^a n^b} + K^2 - K^{ab} K_{ab})$$

$$= \frac{1}{2} R' + \frac{1}{2} K^2 - \frac{1}{2} K^{ab} K_{ab}$$

$T_{ab} n^a n^b = \rho$ [matter/energy density as measured by an observer with 4-velocity $(n^0, n^a)$]

$$\therefore \frac{1}{2}\left(R' - K^{ab} K_{ab} + K^2\right) = 8\pi\rho$$

$$\Rightarrow \boxed{R' - K^{ab} K_{ab} + (K^a{}_a)^2 = 16\pi\rho}$$

acts as a constraint on $h_{ab}$ (and $K_{ab}$).

Now, consider the normal-tangential component:

$$8\pi h_a{}^b T_{bc} n^c = h_a{}^b G_{bc} n^c$$
$$= h_a{}^b R_{bc} n^c$$

Recall: $D_a K^a{}_b - D_b K = h^c_b R_{cd} n^d$

$$\therefore \boxed{8\pi h_a{}^b T_{bc} n^c = D_b K^b_a - D_a K}$$

Since these two equations describe how $K_{ab}$ "evolves" wrt $\mathcal{L}_n$, these are the equations which tell you how Einstein's eq on $\Sigma$ evolve.

Initial value/ data for Einstein's field equation on $M$. $= (\Sigma_1, h_{ab}, K_{ab})$; $\Sigma_1$ is a (Cauchy) hypersurface, $h_{ab} = g_{ab}\big|_{\Sigma_1}$, $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}$ such that the two constraint equations are satisfied.

<u>Theorem</u>: Given $(\Sigma_1, h_{ab}, K_{ab})$ satisfying the 2 constraint equations. Upto isomorphisms, $\exists$ a unique $(M, g_{ab})$ such that $(M, g_{ab})$ satisfies the vacuum Einstein equations, $M$ is globally hyperbolic. (if $\Sigma_1$ is Cauchy, $M$ is inextendible).

We demand that $\Sigma_1$ is Cauchy because otherwise initial data can be singular and lead to "Cauchy horizons". This is done by demanding that $\Sigma_1$ at large $\Delta x_i$ looks like a hypersurface in Minkowski space.

"initial data is asymptotically flat"

Now, we have assumed that the surface is inextendible. But what if you calculate and it turns out that it is extendible?

Conjecture (Penrose): if $(\Sigma, h_{ab}, K_{ab})$ is a geodesically complete, asy. flat initial data for vacuum Einstein eq. Then $(M, g_{ab})$ is inextendible.

Strong cosmic censorship conjecture.

Remarks: (i) For Kerr Black holes, this has been disproven by Dafermos, Luk.

(ii) Weak CCC: No singularities are naked i.e all singularities must be behind a horizon.

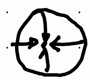
(iii) Weak CCC and Strong CCC are independent of each other.

$\boxed{\text{GR is a fully predictable theory}}$

# Singularity Theorems:

So far, we have seen $(\Sigma, h_{ab}, K_{ab})$

a. Cauchy data on an appropriate H.S $\Sigma_1$ reproduces $(M, g_{\mu\nu}, G_{\mu\nu})$.

b. $g_{\mu\nu}$ for spherically symmetric stars is singular: 

Are singularities a consequence of this spherical symmetry (which can be broken) by radial perturbation, or are they a generic feature in GR?

**Def:** A null hypersurface is a HS whose normals are all null.

Ex: The EH of a schwarzschild BH is a NH.

$n = dr$

$$n^2 = g^{\mu\nu} n_\mu n_\nu = g^{rr} = \left(1 - \frac{2M}{r}\right)$$

$$\Rightarrow \boxed{r = 2M}$$

Let $n_a$ be normal to a NH $N$. For any non-zero vector $X^a$ tangent to $N$, $n_a X^a = 0$

Let $P$ be a smooth null vector field (field of null vectors)

A smooth curve $\gamma : I \to N$ an integral curve of $n \in P$ if $\forall\, t \in I$, [$I$ is an interval in $\mathbb{R}$)

$$\dot{\gamma}(t) = n_{\gamma(t)}$$

Integral curves of $n^a$ are null geodesics.

We refer to them as generators of $N$.

Proof: Given $N$ as $f =$ constant

$$df \neq 0 \text{ on } N. \quad n = h\, df.$$

$\tilde{N} = df$. Since $N =$ null,

$$\tilde{N}^a \tilde{N}_a = 0 \text{ on } N.$$

So $\nabla_a (\tilde{N}^b \tilde{N}_b)\big|_N = 2 \tilde{N}_a \alpha$ → some function

Symmetry of indices: $\nabla_a \tilde{N}_b = \nabla_a \nabla_b f$

$$= \nabla_b \nabla_a f = \nabla_b \tilde{N}_a.$$

$$\cancel{2} \tilde{N}^b \nabla_a \tilde{N}_b \Big|_N = \cancel{2} \alpha N_a \rightarrow$$ geodesic eq with non-affine connection.

Geodesic deviation : $X^\mu = \dfrac{\partial x^\mu}{\partial \tau}$

$x^\mu = (\tau, s)$

$Y^\mu = \dfrac{\partial x^\mu}{\partial s}$

$$\boxed{\frac{D^2 X^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma} X^\nu Y^\rho X^\sigma}$$

Consider an open subset of $M$. The geodesic congruence of $U$ is a family of geodesics in $M$ such that only one geodesic passes through each point $p$ in $U$.

All geodesics are of the same type in the cases we will consider.

Let us consider the case of null geodesics:

Consider a set of geodesics and a HS $\Sigma_1$ that intersects them only once. Let $N^a$ be a null vector field on $\Sigma_1$. Extend $N^a$ from $\Sigma_1$ along the geodesics ($N^2 = 0$, $U \cdot N = -1$, $U \cdot \nabla^a N_a = 0$)

The deviation vector:

$$S^a = \alpha U^a + \underbrace{\beta N^a}_{} + \underbrace{\hat{S}^a}_{},$$

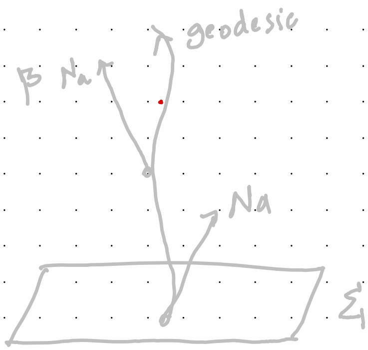
Orthogonal to $U$

parallel trasported along a geodesic.

$$\hat{S}^a \Rightarrow U \cdot \hat{S} = N \cdot \hat{S} = 0.$$

↓

spacelike or zero.




$\hat{S} = \nabla N^a$

A deviation vector for which $U \cdot S = 0$

satisfies $U \cdot \nabla \hat{S}^a = \hat{B}^a{}_b \hat{S}^b$

$$\hat{B}^a{}_b = P^a{}_c B^c{}_d P^d{}_b,$$

$$P^a{}_b = \delta^a_b + N^{\{a}{}_, U_{b\}}$$

↓

Projection operaton which projects tangent space.

Now, we can define expansion/shear/rotation of geodesics

$\theta$ = expansion = $\hat{B}^a{}_a$ (trace)

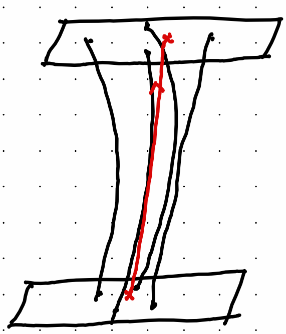
$\hat{\sigma}_{ab} = \hat{B}_{(ab)} - \frac{1}{2} P_{ab}\theta$ (traceless-symm) shear

$\hat{\omega}_{ab} = \hat{B}_{[ab]}$ [antisymm] rotation.

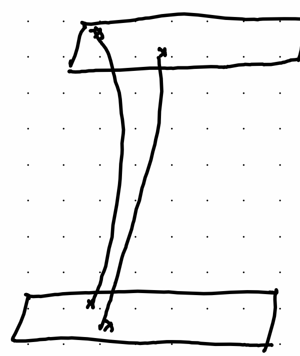
$$\hat{B}^a{}_b = \frac{1}{2}\theta P^a{}_b + \hat{\sigma}^a{}_b + \hat{\omega}^a{}_b.$$

in fact $\theta = g^{ab} B_{ab} = \nabla_a u^a$ (exercise)

if the geodesic congruence contains generators of $N$, then $\hat{\omega}_{ab} = 0$.

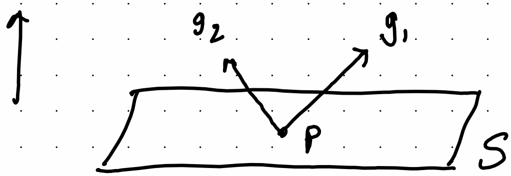


expansion
(more apart in every dir)



shear
(more apart in 1 direction)

Consider a 2d space like surface (all tangent vectors are space like).




two future directed nullvectors orthogonal to $S$.

We have 2 families of null geodesics which start on $S$. So we can form 2 null hypersurfaces from $g_1, g_2$. ($N_1$, $N_2$) Let $\omega_{ab} = 0$ on $N_1$ and $N_2$.

<u>Def</u>: $S$ is said to be a trapping surface if $\theta$ for both $N_1$ and $N_2$ are negative.

$S$ is marginally trapped if $\theta$ for $N_1, N_2$ $\leq 0$.

As you vary along a geodesic, you can compute how $\theta$ changes.

$$\boxed{\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 + \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - R_{ab}U^aU_b}$$

This is the famous Raychandhui equation.

(Deriving this is a great exercise in GR).

(We can derive this during office hours if you want)

Since the Raychaudhuri equation involves a contraction of $R_{ab}$ with a velocity (tangent) vector, (if $R=0$), this implies contraction with the stress-energy tensor. So this constrains how we apply the Raychaudhuri equation:

a. $-T^a{}_b V^b$ is either zero or future directed causal, $\forall$ timelike future directed $V^a$.

Dominant EC.

b. Weak EC: $T_{ab} V^a V^b \geq 0$ for any causal vector

c. Null EC: $T_{ab} V^a V^b \geq 0$ for any null vector

(No tachyons / geometry that allows for tachyons)

Strong EC: $\left(T_{ab} - \frac{1}{2} g_{ab} T^c{}_c\right) V^a V^b \geq 0$

"Gravity is attractive"

These EC's are indep. SEC is violated in $\Lambda > 0$.

- if we have geodesics satisfying NEC,

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2. \quad \text{(see RE)}$$

- if $\theta < 0$ ($\theta = \theta_0$, $\theta_0 < 0$) at a point $p$ on $\gamma$, a curve that generates a NH. Then, if $\gamma$ can be extended in affine param [i.e if $\gamma$ can be complete wrt a geodesic), then $\theta < -\infty$ within an affine param distance $2/\theta_0$.

<u>Proof</u>: $\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \Rightarrow \frac{d\theta^{-1}}{d\lambda} \geq \frac{1}{2}$

$$\int_{\theta}^{\theta_0} \frac{1}{\theta^2}\, d\theta \leq -\frac{1}{2}\lambda \quad \Rightarrow \quad \theta \leq \frac{\theta_0}{1 + \lambda\theta_0/2}$$

if $\theta_0 < 0$, $\lim_{\lambda \to 2/|\theta_0|} \theta = -\infty$.

if the geodesic deviation equation vanishes at 2 points $p, q$ along the geodesic, then $p, q$ are conjugate.

Causal structure of space time:

Def: Let $(M, g)$ be a time-orientable spacetime, Let $U \subset M$. The chronological future of $U$, $I^+(U)$, is the set of all points on $M$ that can be reached by a future-directed timelike curve starting on $U$.

The causal future of $U$, $J^+(U) =$

$$U \cup \{p \in M \text{ st, } \forall\, m \in U,\ p, m \text{ lie on a future directed causal curve } \gamma\}.$$

$I^-(U)$ and $J^-(U)$ are defined similarly but for past directed.

Ex: Let $q \in \text{Mink}_{1,d-1}$.

$$L^+(q) =: I^+(q)$$

$$J^+(q) = L^+(q) \cup \{q\}.$$

<u>Def</u>: The future Cauchy horizon of a partial cauchy surface $\Sigma_1$ is

$$H^+(\Sigma_1) = \overline{D^+(\Sigma_1)} \setminus I^-(D^+(\Sigma_1))$$

past CH: $H^-(\Sigma_1) = \overline{D^-(\Sigma_1)} \setminus I^-(D^-(\Sigma_1))$

a surface on which you can no longer have predictability in terms of Cauchy data.

<u>Singularity thm</u>: Let $(M, g)$ be a globally hyperbolic spacetime with a non compact cauchy surface $\Sigma_1$. Let the Einstein eq and NEC be satisfied. $M$ contains a trapped surface Let $\theta_0 < 0$ be the max value of $\theta$ on $T$ for both sets of null geodesics orthogonal to $T$. At least one of these geodesics is future inextendible and has affine length $\leq 2/|\theta_0|$.

Singularities are inevitable the moment a geodesic congruence has negative expansion. These trapped surfaces lie just under the horizon and are dynamically formed in grav collapse.

Regardless of spacetime, ... singularities can be formed generically in gravity.

## Penrose – Carter diagrams:

where is the $\infty$ in asymptotic infinity?
It is possible to "bring" this $\infty$ into spacetime by conformal compactification. (This does not change the causal structure of spacetime).

All points that are $\infty$ly far away in proper distance are only finitely far away in terms of the affine parameter of a new metric.

Example: $\text{Mink}_{1,d-1}$, $d=4$

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$$

$$\left\{ \begin{array}{l} u = t-r \\ v = t+r \end{array} \right\} \longrightarrow \quad ds^2 = -du\,dv + \frac{(u-v)^2}{4} d\Omega_2^2$$

set $\quad u = \tan U$, $\quad V, U \in (-\pi/2, \pi/2)$

$\quad v = \tan V$

$V > U$.

$$ds^2 = (2\cos U \cos V)^{-2} \left[-4\,dU\,dV + \sin^2(V-U) d\Omega_2^2\right]$$

if either/and $U, V = \frac{\pi}{2}$, we get $ds^2 = \infty$.

$\Lambda = 2\cos U \sin U$.

$$d\tilde{s}^2 = \Lambda^2 ds^2 = -4\,dU\,dV + \sin^2(V-U)\, d\Omega_2^2.$$

In this new conformally scaled metric, we can add points at $\infty$. $V \geq U$

$$\left.\begin{array}{l} U = -\frac{\pi}{2} \\ V = \frac{\pi}{2} \end{array}\right\} \longrightarrow \begin{array}{l} u = -\infty \\ v = \infty \end{array} \longrightarrow \begin{array}{l} r \to \infty \\ t \text{ finite} \end{array} : \begin{array}{l} \text{Spatial } \infty \\ i_0 \end{array}$$

$$\left.\begin{array}{l} U = \pm\frac{\pi}{2} \\ V = \pm\frac{\pi}{2} \end{array}\right\} \longrightarrow \begin{array}{l} u = \pm\infty \\ v = \pm\infty \end{array} \longrightarrow \begin{array}{l} t \to \pm\infty \\ r \text{ finite.} \end{array} : \begin{array}{l} \text{past/future temporal } \infty \\ i_\pm . \end{array}$$

$$\begin{array}{l} U = -\frac{\pi}{2} \\ |V| \neq \pi/2 \end{array} \longrightarrow \begin{array}{l} u = -\infty \\ v = \text{finite.} \end{array} \rightarrow \begin{array}{l} r \to \infty \\ t \to -\infty \\ r + t \text{ finite} \end{array} \rightarrow \begin{array}{l} \text{Past null } \infty . \\ \mathscr{I}^- \end{array}$$

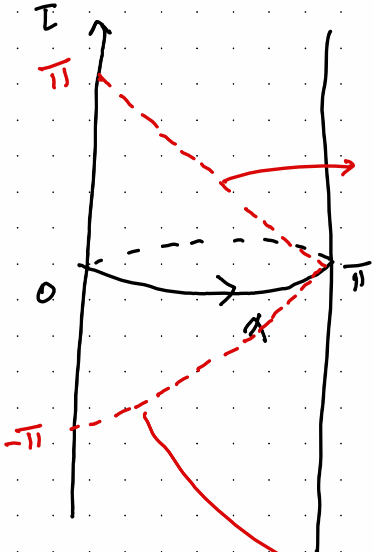
$$\begin{array}{l} |U| \neq \frac{\pi}{2} \\ \tilde{V} = \frac{\pi}{2} \end{array} \longrightarrow \begin{array}{l} u \text{ finite} \\ v = \infty \end{array} \longrightarrow \begin{array}{l} r \to \infty \\ t \to \infty \\ r - t \text{ finite} \end{array} \longrightarrow \begin{array}{l} \text{Future null } \infty \\ \mathscr{I}^+ \end{array}$$

$ds^2_{\text{mink}} \hookrightarrow d\tilde{s}^2$, boundary at $\Lambda = 0$

Let $\tau = V + U$, $\chi = V - U$

$$d\tilde{s}^2 = -d\tau^2 + d\chi^2 + \sin^2\chi \, d\Omega_2^2$$

$$\Lambda = \cos\tau + \cos\chi . \qquad \chi \sim \chi + 2\pi$$




if each of the 2 sphere at const $\chi$ is a point,

$\chi + \tau = \pi$, $v = \pi/2$, $\mathcal{J}^+$

Mink:

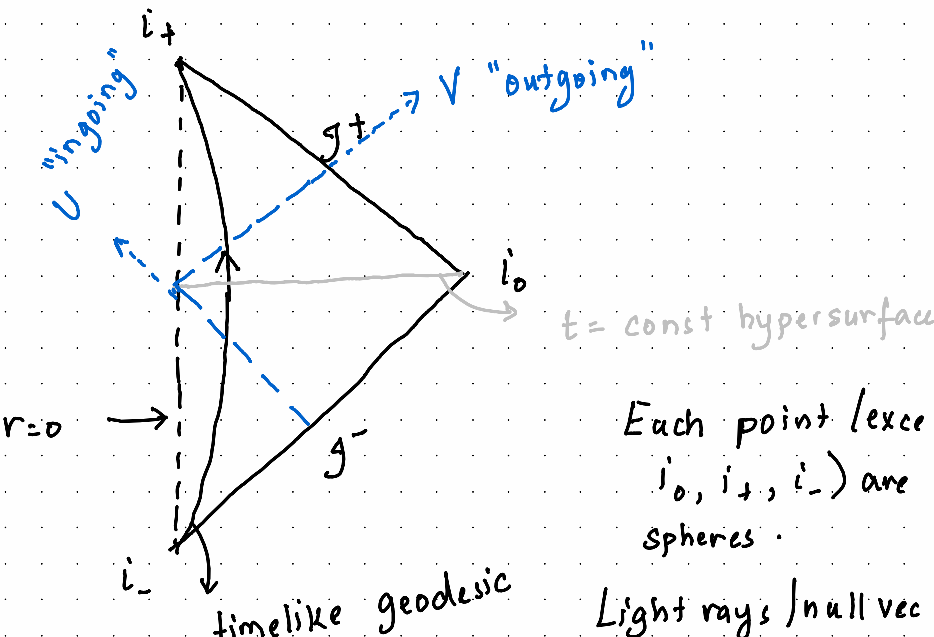
$-\pi < \tau < \pi$

$0 \leq \chi \leq \pi$

$-\pi < \chi + \tau \leq 2\pi$

$\chi - \tau = \pi$, $\tilde{u} = -\frac{\pi}{2}$, $\mathcal{J}^-$

Now, squash the cylinder.




Each point (except $i_0$, $i_+$, $i_-$) are 2-spheres.

Light rays / null vec ($\mathcal{J}^- \to \mathcal{J}^+$)

massive $\to$ $i_- \to i_+$.

Example: Kruskal spacetime:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv + r^2 d\Omega_2^2 \qquad (I)$$

$u = \tan U \,, \qquad U \in (-\pi/2, \pi/2)$

$v = \tan V \qquad V \in (-\pi/2, \pi/2)$

$$ds^2 = (2\cos V \cos U)^{-2} \left[ -4\left(1 - \frac{2M}{r}\right) dU\, dV + r^2 \cos^2 U \cos^2 V \, d\Omega_2^2 \right]$$

$$2r^* = v - u \Rightarrow r^* = \frac{\tan V - \tan U}{2} = \frac{\sin(V-U)}{2\cos V \cos U}.$$

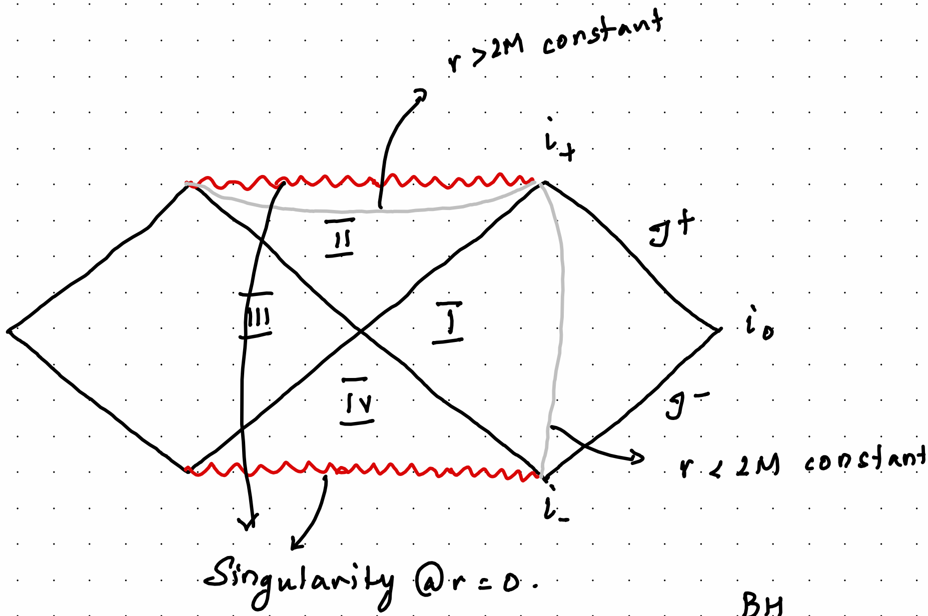
$$r^* = r + 2M \ln\left(\frac{r-2M}{2M}\right)$$

$$d\tilde{s}^2 = \Lambda^2 ds^2 = -4\left(1 - \frac{2M}{r}\right) dU\, dV + \left(\frac{r}{r_*}\right)^2 \sin^2(V-U) \, d\Omega_2^2$$

$$\lim_{r \to \infty} d\tilde{s}^2 = -4\, dU\, dV + \sin^2(V-u)\, d\Omega_2^2$$

which is $ds^2_{Mink}$.

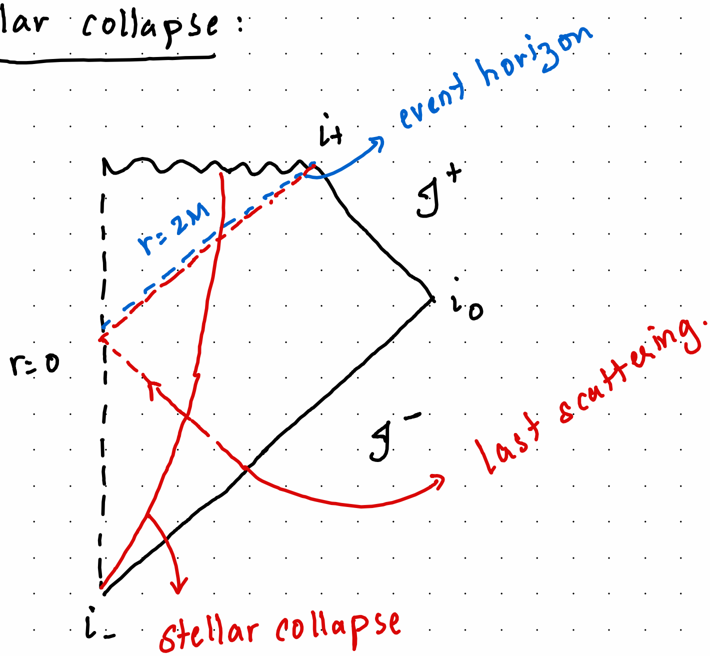
Kruskal is "asymptotically flat"! So we can add $\mathcal{J}^{\pm}$.




"Penrose diagram for a Kruskal BH spacetime"

All r = const hypersurfaces meet @ $i_+$.

## Stellar collapse:



Let $M$ be an asymptotically flat spacetime.

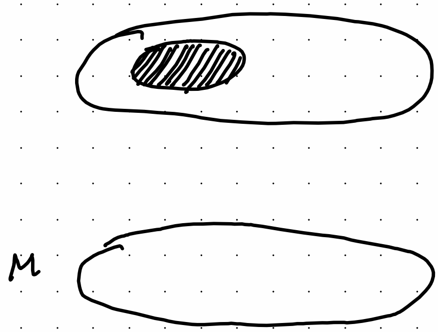
Let $U \subset M$. $J^-(U)$ is the causal past of $U$.

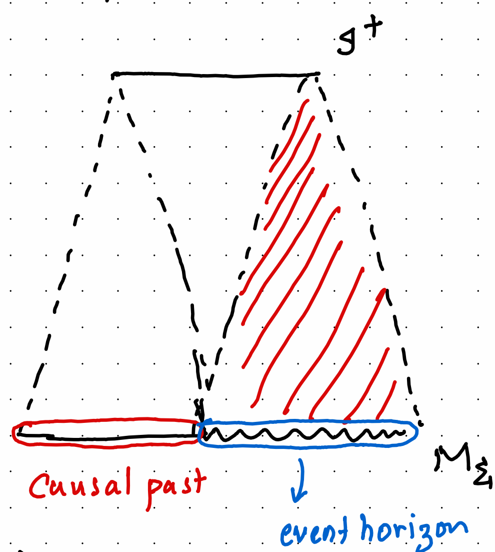
$\bar{J}^-(U)$ = closure of $J^-$ including limit points
(Union of $U$ and its Bdy)

$\partial \bar{J}^-(U) := \dot{J}^-(U) = \bar{J}^-(U) - J^-(U)$

Future event horizon of $M$ is

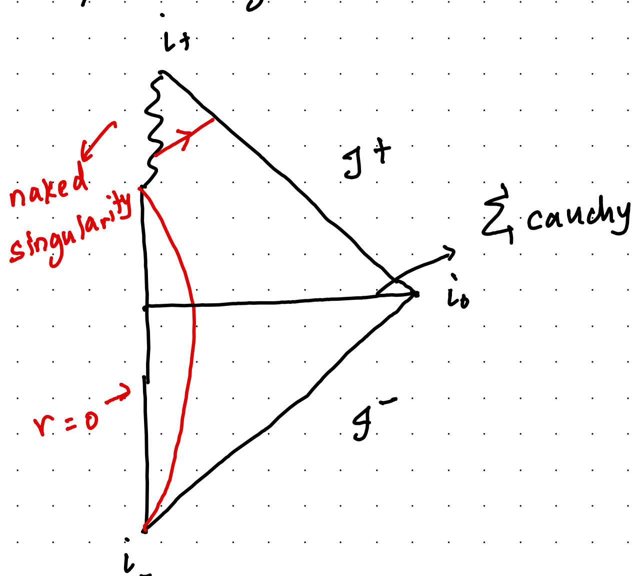
$\mathcal{H}^+ = \dot{J}^-(\mathscr{I}^+)$ i.e bdy of closure of causal past of $\mathscr{I}^+$.




- $i_0, \mathscr{I}^- \not\subset \mathcal{H}^+$
- $\mathcal{H}^+$ is a hypersurface (null).
- No two points on $\mathcal{H}^+$ are timelike separated
- generators of $\mathcal{H}^+$ may have past end points
- generators of $\mathcal{H}^+$ have no future end points (singularity thm)

The past singularity in Kruskal is naked.




if this were possible, then $(h_{ab}, K_{ab}, \Sigma_{cauchy})$ cannot predict $(g_{ab}, G_{ab})$ on $\mathcal{J}^+$.

Cosmic Censorship Conjecture:

Naked singularities cannot form in asy. flat M by grav collapse if $\Sigma(t)$ is non-singular for some $t$.

Unsolved!

# Reissner - Nordström black holes

These are black holes charged under a $U(1)_{EM}$ field.

$$S = \frac{1}{2K} \int d^4x \sqrt{-g} \left[ R - F_{\mu\nu} F^{\mu\nu} \right]$$

Unusual normalization of the Maxwell term.

$$\left.\begin{aligned} G_{\mu\nu} &= 2\left( F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \\ D_\mu F^{\mu\nu} &= 0 \end{aligned}\right\} \text{source free}$$

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)} + r^2 d\Omega_2^2$$

actually $Q \sim Q(M)$

$Q(M=0) = 0$ (in fact we will see that this relevant)

$$A = \frac{Q}{r} dt \quad (F = dA, \text{ 1- form Maxwell potential})$$

Birkhoff thm: generalization for EM equation

$$ds^2_{RN} = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\Omega_2^2$$

$$\Delta = r^2 - 2Mr + Q^2$$

$\Delta = 0 \longrightarrow r_\pm = M \pm \sqrt{M^2 - Q^2}$

$M < |Q|$, then we have naked singularities X

<u>HW</u>: Consider a collapsing shell of matter, mass $M$, charge $Q$. Consider total energy/mass as a function of $R$. Show that the collapsing shell is possible iff $M > Q$.

Sort of obvious:

$$\frac{GM^2}{R} = \frac{GQ^2}{R} \longrightarrow$$ Coulomb potential is holding the grav pot. in eq.

To analyze $M > |Q|$, we introduce the EF coords

For $r > r_+$ : $$dr_* = \frac{r^2}{\Delta} dr$$

$$r_* = r + \frac{1}{2K_+} \ln \left| \frac{r - r_+}{r_+} \right| + \frac{1}{2K_+} \ln \left| \frac{r - r_-}{r_-} \right| + \text{Const}$$

(check this)

where $$K_\pm = \frac{(r_\pm - r_\mp)}{2r_\pm^2}$$

These are the inner and outer surface gravities i.e acceleration of a particle at the horizon, as measured by an observer at $\infty$

Now, $u = t - r_*$, $v = t + r_*$

(Ingoing EF coordinates)

$$ds^2 = -\frac{\Delta}{r^2} dv^2 + 2 dv dr + r^2 d\Omega_2^2$$

This metric is smooth for $r > 0$ and can therefore be continued to $r < r_+$.

$r_+$ : Outer event horizon
$r_-$ : inner event horizon
} both null hypersurfaces

(analogously, using outgoing EF coordinates)

$$ds^2 = -\frac{\Delta}{r^2} du^2 - 2 du dr + r^2 d\Omega_2^2$$

(white hole of RN solution)

For the case: $M = |Q|$, the two horizons coincide $r_+ = r_-$ and $K_+ = K_- = 0$. This means that the RN solution is extremal and has no surface grav.

Also, $T = \frac{\kappa}{2\pi}$ (Temp $\propto$ surface gravity)

So extremal solutions have zero temperature.

Kruskal - coordinates for RN black holes:

We want to understand the global structure of the Reissner-Nordström solution. So we introduce Kruskal like coordinates:

$$U^{\pm} = -e^{-\kappa_{\pm} u}, \quad V^{\pm} = \pm e^{\kappa_{\pm} v}$$

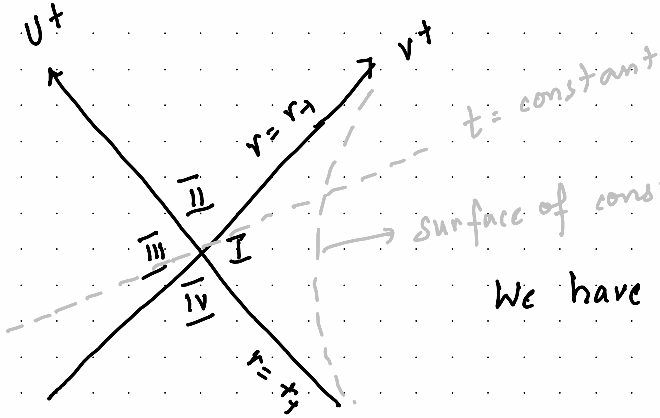
Start in $r > r_+$. $(v^+, u^+, \theta, \phi)$

$$ds^2 = -\frac{\Delta}{r^2} dr^2 + 2\, dv\, dr + r^2 d\Omega_2^2$$

HW:
$$ds^2 = -\frac{r_+ r_-}{\kappa_+^2 r^2} e^{-2\kappa_+ r} \left(\frac{r - r_-}{r_-}\right)^{1 + \frac{\kappa_+}{|\kappa_-|}} dU^+ dV^+$$
$$+ r^2 d\Omega_2^2$$

$r(u^+, v^+) \Rightarrow$

$$-U^+ V^+ = e^{2\kappa_+ r} \left(\frac{r - r_+}{r_+}\right) \left(\frac{r_-}{r - r_-}\right)^{\frac{\kappa_+}{|\kappa_-|}}$$




(ER bridge)

We have $U^+ < 0$, $V^+ > 0$ $\Rightarrow r > r_+$

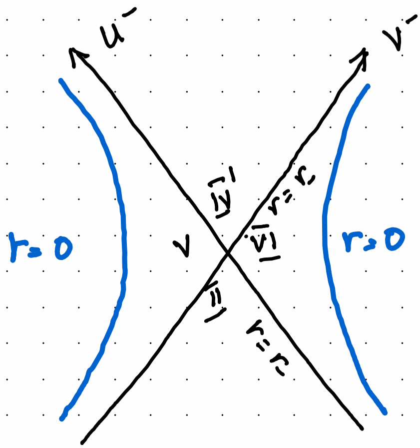
No singularities in $\overline{\text{II}}$ an $\overline{\text{IV}}$ because $r(u^+, v^+) > r_-$. if we continue to $u^+ \geq 0$ and $v^+ \leq 0$, we will see singularities.

$u^+ = v^+ = 0$: Null hypersurfaces intersect at a bifurcation 2-sphere.

To investigate the RN metric at $r \leq r_-$, define the retarded coordinates: $v = t + r_*$, $u = t - r_*$ and use $U^-$, $V^-$ as before:

This now gives $U^-, V^- < 0$ in $\overline{\text{II}}$:

$$ds^2 = \frac{-r_+ r}{\kappa_-^2 r^2} e^{2|\kappa_-| r} \left(\frac{r_+ - r}{r_+}\right)^{1 + |\kappa_-|/\kappa_+} dU^- dV^- + r^2 d\Omega^2$$




$$U^- V^- = e^{-2|\kappa_-| r} \left(\frac{r-r_-}{r_-}\right) * \left(\frac{r_+}{r_+ - r}\right)^{|\kappa_-|/\kappa_+}$$

We have new regions V and VI $0 < r < r_-$ which contain curvature singularities at $r=0$ ($U^- V^- = -1$). IV' ~ IV by isometry and can be continued to new regions I', II', III'.. I' and III' are new regions (asy. flat) isometric to I and III.

RN solutions are fascinating:

Consider a path of constant $r, \theta, \phi$ for a region where $\Delta < 0$. Ex (II)

$$ds^2 = -\frac{\Delta}{r^2} dv^2 \quad \text{(IEFC)}$$

$$= +\frac{|\Delta|}{r^2} dv^2 \longrightarrow \text{space like}$$

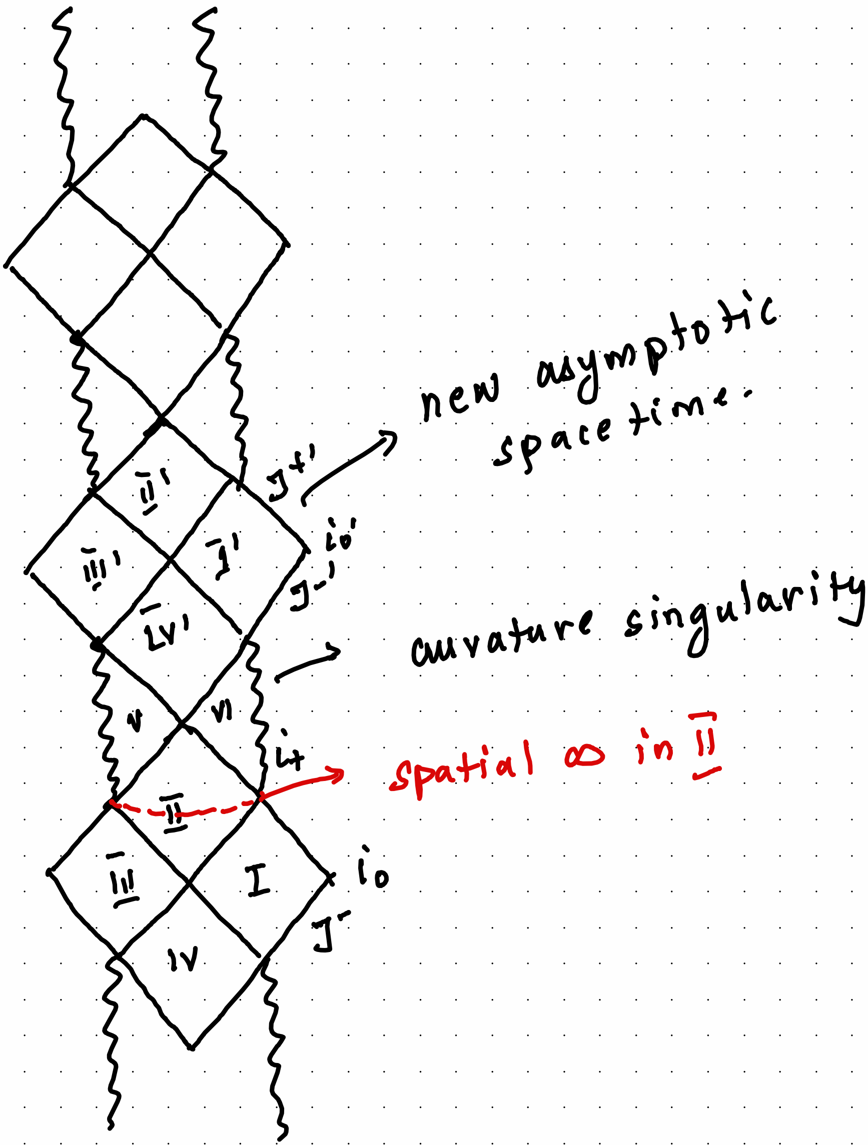
The proper length from $v=0$ to $v=-\infty$

$$\Rightarrow S = \int_{-\infty}^{0} \frac{|\Delta|^{1/2}}{r} dv = \frac{|\Delta|^{1/2}}{r} \int_{-\infty}^{0} dv = \infty .$$

Behind $r=r_+$ in $\bar{II}$, $\exists$ a spatial $\infty$.

(Since $V^{\pm}=0$ can be reached in finite proper time, these hypersurfaces are apart of spacetime)

At first glance it seems that ccc is violated here.



new asymptotic spacetime.
curvature singularity
spatial ∞ in II
II'
III'
I'
IV'
$\mathcal{J}^{+'}$
$i_0'$
$\mathcal{J}^{-'}$
V
VI
$i_+$
II
III
I
IV
$i_0$
$\mathcal{J}^-$

The RN solution is actually unstable:

Let A cross the EH and enter $\underline{\overline{II}}$ and hovers there.
B stays in $\underline{\overline{I}}$. B sends a pulse of energy at regular intervals. A recieves these signals in finite time. These signals get blue shifted so the $T_\mu{}^\mu$ content in $\underline{\overline{II}}$ grows. A tiny pulse) perturbation in $\underline{\overline{I}}$ gets amplified in $\underline{\overline{II}}$.
The effect of this instability is that the Cauchy HS at spatial $\infty$ collapses to a horizon.
i.e RS $\longrightarrow$ Schwarzschild.

(BHs quickly neutralize themselves by plucking out an electron/anti-electron from the vacuum).

$\downarrow$

RN non-extremal

Let's look at extremal RN black holes:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

$M = |Q|$

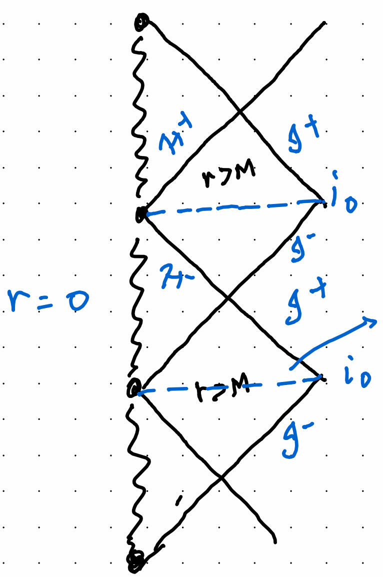
$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2$$

$$dr_* = \frac{dr}{\left(1 - \frac{M}{r}\right)^2} \rightarrow r_* = r + 2M \ln\left|\frac{r-M}{M}\right| - \frac{M^2}{r-M}.$$

$u = t - r_*$, $v = t + r_*$

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2\,dv\,dr + r^2 d\Omega_2^2$$

which is smooth and extendible until $r = 0$.




constant $t$ hypersurfaces are no longer ER bridges. but rather an $\infty$ throat b/w $r = 0$ and spatial $\infty$.

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2$$

$$r = M(1+\lambda) \rightsquigarrow \quad \frac{M}{r} = \frac{1}{1+\lambda}$$

$$dr = M\, d\lambda$$

$$ds^2 = -\left(1 - \frac{1}{1+\lambda}\right)^2 dt^2 + \left(1 - \frac{1}{1+\lambda}\right)^{-2} dr^2 + r^2 d\Omega_2^2$$

$$\simeq \underbrace{-\lambda^2 dt^2 + \frac{M^2}{\lambda^2} d\lambda^2}_{AdS_2} + \underbrace{M^2 d\Omega_2^2}_{S_2}$$

NH of extremal RN BHs $\simeq AdS_2 \times S^2$

(Bertotti - Robinson)

A Key result in AdS/CFT applications!

Can also introduce new radial coordinates

$$\rho = r - M$$

$$ds^2_{\text{Ext-RN}} = -H^{-2}dt^2 + H^2\left(d\rho^2 + \rho^2 d\Omega_2^2\right)$$

$$H = 1 + \frac{M}{\rho}.$$

special case

$$ds^2 = -H(x)^{-2}dt^2 + H(x)^2\left(dx^2 + dy^2 + dz^2\right)$$

$$\nabla^2 H = 0 \quad : \quad H = 1 + \sum_{i=N}^{N} \frac{M_i}{|x - x_i|}$$

3d Laplacian

N-RN BH's of mass $M_i$, $|Q_i| = M_i$

at $x_i$.

"Multi center Black holes"

RN : → stability of spacetime

→ weak gravity conjecture ...

# The Kerr and Kerr-Newman spacetimes

**Def**: A spacetime asymptotically flat at null infinity (i.e asymptotically Minkowski) is stationary if it admits a Killing vector field that is timelike in a neighbourhood of $\mathcal{J}^{\pm}$. if $K^a$ is hypersurface orthogonal, it is static.

**Def**: if $K^a$ above is spacelike near $\mathcal{J}^{\pm}$ and if it generates a 1 parameter group of isometries isomorphic to $U(1)$, $M$ is axisymmetric.

**Theorem (Israel)**: if $(M, g)$ is a static BH spacetime (asy flat), $M$ is Schwarzschild upto isometries.

**Theorem (Hawking; Wald)**: if $(M, g)$ is a stationary, non-static, asy flat solution of Einstein-Maxwell action, then $(M, g)$ is axisymmetric.

**Theorem** (Carter): if $(M, g)$ is a stationary axisymmetric, asy flat vacuum solution outside a BH, then $(M, g)$ is a 2-parameter family of solution; params: $M, J$.

Generically: BH's in the universe are Kerr.

Extendible to 4 param $(M, Q, P, J)$

"Kerr-Newman:

**Derivation ?**

**Kerr-Newman solution:** $(M, Q, P, J)$

$$ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma_1} dt^2 - \frac{2a \sin^2\theta (r^2+a^2-\Delta)}{\Sigma_1} * $$

$$* \, dt \, d\phi + \left( \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma_1} \right) \sin^2\theta \, d\phi^2$$

$$+ \frac{\Sigma_1}{\Delta} dr^2 + \Sigma_1 d\theta^2$$

"Boyer-Lindquist" coords

$$A = -\frac{Qr}{\Sigma_1}\left(dt - a\sin^2\theta\, d\phi\right) + \frac{P\cos\theta}{\Sigma_1} *$$

$$* \left(a\,dt - (r^2+a^2)\,d\phi\right)$$

$$\Sigma_1 = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2 + e^2$$

$$e = \sqrt{Q^2+P^2}$$

$$\lim_{a\to 0} \left(ds^2, A\right)_{KN} \longrightarrow \left(ds^2, A\right)_{RN} \quad ✓$$

$$\lim_{\substack{a\to 0\\ P,Q\to 0}} (KN) \longrightarrow \text{Schwarzschild.} \quad ✓$$

$$\lim_{P,Q\to 0} (KN) \longrightarrow \text{Kerr.}$$

Kerr Solution :

Set $e = 0$ in the Kerr - Newman metric

Kerr metric : in Boyer - Lindquist coords:

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma} dt\,d\phi$$

$$+ \frac{\left((r^2+a^2)^2 - \Delta a^2\sin^2\theta\right)}{\Sigma}\sin^2\theta\, d\phi^2$$

$$+ \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 ,$$

$$a := \frac{J}{M}$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

$$\Delta = (r + r_-)(r + r_+)$$

$$r_\pm = M \pm (M^2 - a^2)$$

① There is no Birkhoff's theorem for Kerr spacetime.

② $g_{tt} = 0 \Rightarrow \Delta = 0, \theta = 0$

Singularity: $\Sigma = 0 \Rightarrow r = 0, \theta = \frac{\pi}{2}$

Consider:

$$r_\pm = M \pm (M^2 - a^2)^{1/2}$$

Case: $M^2 < a^2$

$\Delta$ has no real roots, but the metric still has singularities $\Rightarrow$ naked! (excluded)

To understand this better: go to Kerr-Schild coordinates:

$$x + iy := (r + ia) \sin\theta \, e^{i\int(d\phi + \frac{a}{\Delta} dr)}$$

$$z := r\cos\theta$$

$$\tilde{t} := \int\left(dt + \frac{r^2 + a^2}{\Delta} dr\right) - r$$

Plug in:

$$r^4 - (x^2+y^2+z^2-a^2)\, r^2 - a^2 z^2 = 0$$

$$ds^2 = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr^3}{r^4+a^2z^2}\left[\frac{r(x\,dx + y\,dy) - a(x\,dy - y\,dx)}{r^2+a^2} + \frac{z\,dz}{r} + d\tilde{t}\right]^2$$

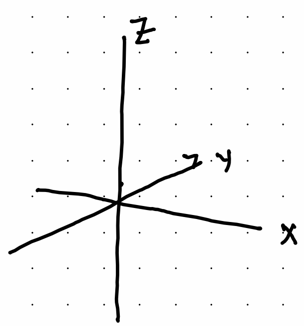
See that Lim $M \to 0$ of above is Minkowski.

Solve for surface of const $\hat{t}$, $r$:

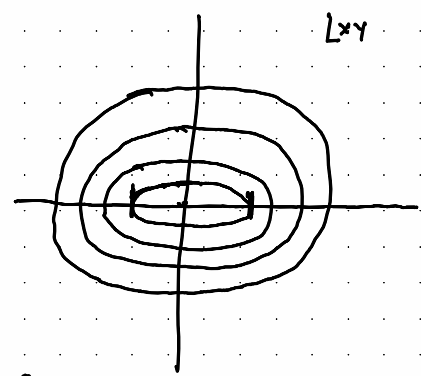
Let $r=0$ be the limit we take $(z=0)$

ie colimit, $r, z \to 0$. degenerates

$$x^2+y^2-a^2=0$$




:



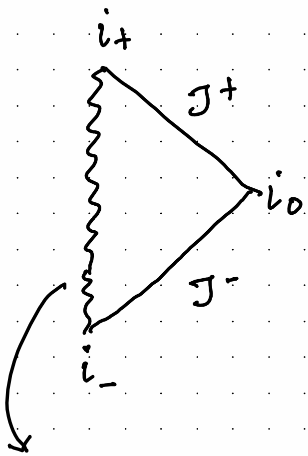

$$x^2+y^2=a^2$$

$\theta = \frac{\pi}{2} \to$ The singularity is the ring

$$x^2+y^2=a^2, \; z=0$$

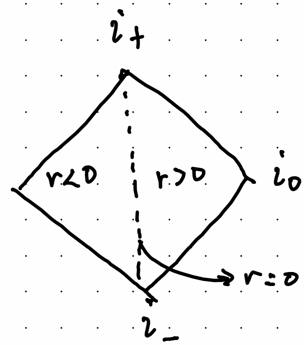
## Causal structure of $M^2 < a^2$:

$\theta = \frac{\pi}{2}$




naked singularity at $r = 0 \Rightarrow$
$x^2 + y^2 = a^2$

$\theta = 0$




if you start @ $r > 0$, you can pass through the ring $x^2 + y^2 = a^2$ at $r = 0$ into $r < 0$.

Also

Notice: $g$ is independent of $\phi$.

$m = \partial_\phi$ is a Killing vector.

$$m^2 = g_{\phi\phi} = a^2 \sin^2\theta \left(1 + \frac{r^2}{a^2}\right) + \frac{M^2 a^2}{r} \left(\frac{2\sin^4\theta}{1 + \frac{a^2}{r^2}\cos^2\theta}\right)$$

Let $r/a = \delta$, $r < 0$, $\delta \ll 1$

$\theta = \pi/2 + \delta$ [you are close to singularity]

$$m^2 \simeq a^2 + \frac{Ma}{\delta} + O(\delta).$$

$$m^2 \simeq a^2 - \frac{Ma}{|\delta|},$$ which $\Rightarrow m^2 < 0$ for small enough $|\delta|$.

So $m^2$ = timelike near ring singularity.

But motion in $\phi$ direction is periodic and $\therefore$ $\partial_\phi$ should have closed orbits.

meaning your spacetime has closed time like orbits. $\downarrow$ CTC's

Global violation of causality.

Case $M^2 > a^2$:

The same ring singularity exists but $r_+, r_-$ are singular radii ie horizons.

These are coordinate singularities:

$$dv := dt + \frac{(r^2 - a^2)}{\Delta} dr$$

(Kerr coords)

$$d\chi := d\phi + \frac{a}{\Delta} dr$$

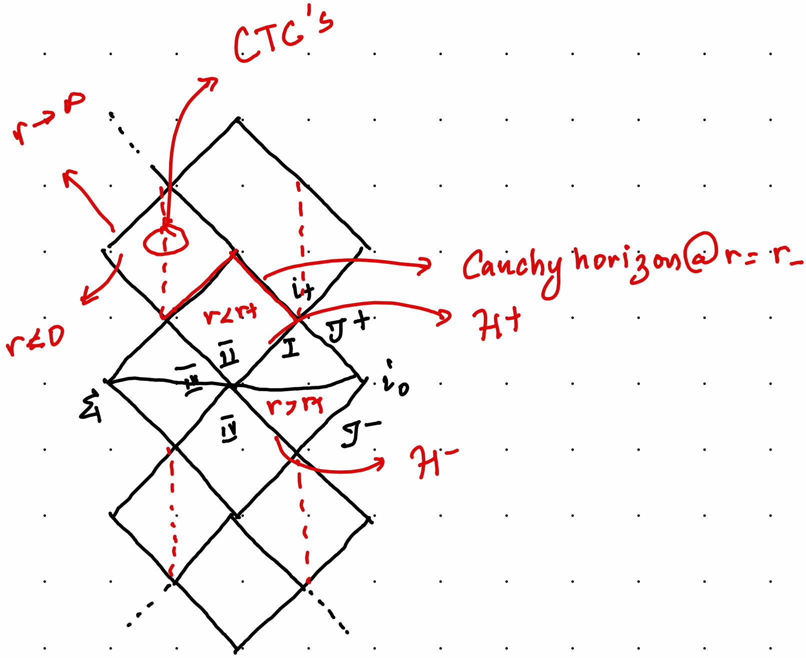
actually disovered first.

Analogous to IEF:

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} dv^2 + 2\,dv\,dr - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma} dv\,d\chi - 2a\sin^2\theta\,d\chi\,dr + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,d\chi^2 + \Sigma\,d\theta^2$$

The singularity at $\Delta = 0$ no longer persists.

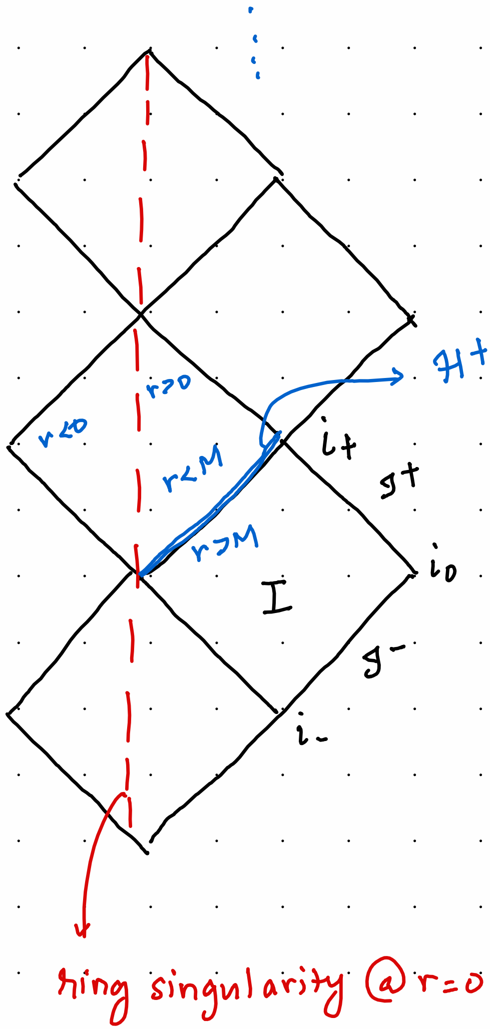
Analogous to RN: $\kappa_\pm = \dfrac{r_\pm - r_\mp}{2(r_\pm^2 + a^2)}$




Penrose - Carter diagram of Kerr.

Case: $M^2 = a^2$

$r_+ = r_-$, $\kappa_\pm = 0$




You could in principle live forever inside an extremal Kerr Black hole.

## The Ergosphere:

$\partial_t$ is a Killing vector of Kerr (stationary)

$$k^2 = g_{tt} = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} = -\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)$$

time like: $1 - \frac{2Mr}{r^2 + a^2\cos^2\theta} > 0$

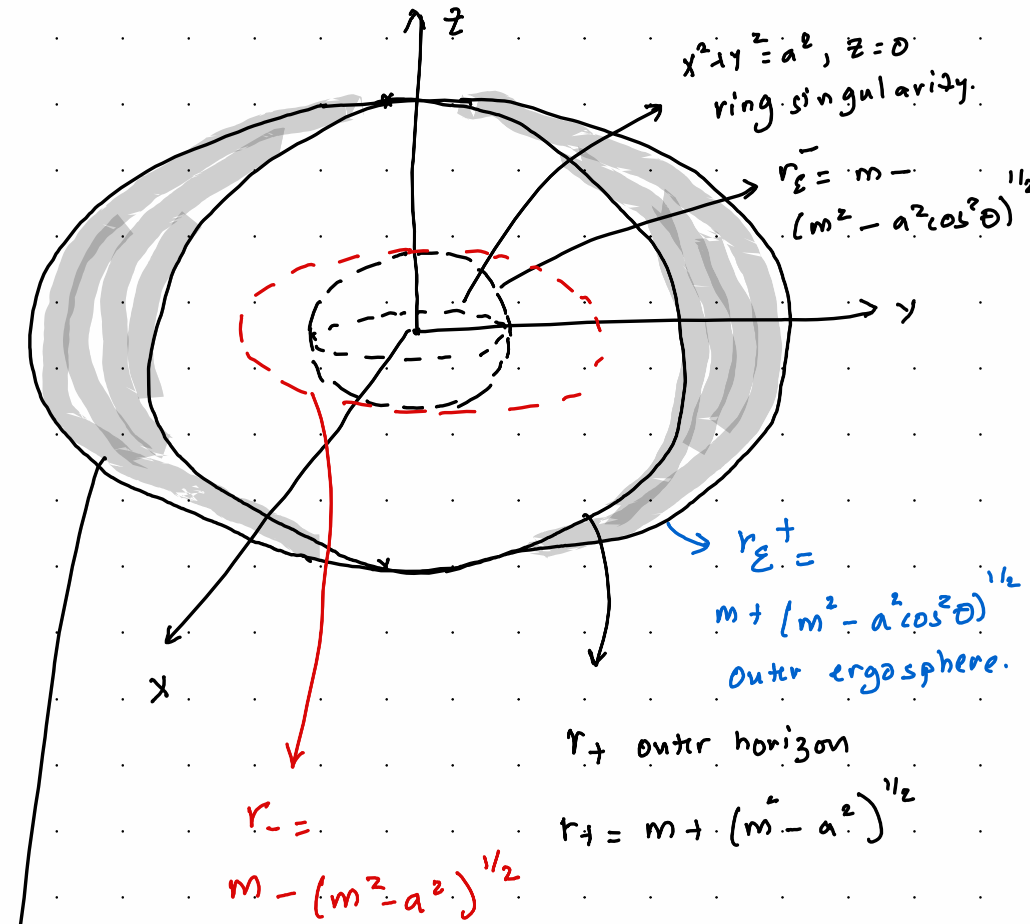
$$\Rightarrow 2Mr < r^2 + a^2\cos^2\theta$$

$$\Rightarrow r^2 + a^2\cos^2\theta - 2Mr > 0$$

$$r_E = M \pm (M^2 - a^2\cos^2\theta)^{1/2}$$

is the hypersurface "ergosphere".

$\theta = 0, \pi,\ r_E = r_+$

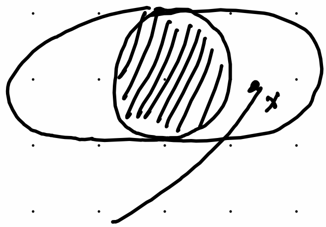



Shaded Region "Ergosphere"

Region where $K$ becomes spacelike outside EH. Meaning that you cannot stay still!

You will "corotate" with the BH.

# Black hole slingshots :




@ X : Fire a propellant in a "counter rotating" orbit.

The Body can now have enough energy to escape the ergoregion with $\Delta E$,

$$\Delta E \leq \begin{cases} 0.21 \, M_{Body} , & \text{Kerr.} \\ 0.29 \, M_{Body} , & \text{Kerr Newman} \end{cases}$$

Let 4-momentum of a body be $p$. It approaches a Kerr BH along a geodesic

$$E = -p \cdot K$$

@ x , $E_1 = -p_1 \cdot K$ , $E_2 = -p_2 \cdot K$
(into BH)

$$E_2 = E - E_1$$
$$= E + p \cdot K.$$

But if this happens inside the ergoregion, $K$ is spacelike

$\therefore$ $p_\mu K^\mu > 0$ !

$\therefore$ $\boxed{E_2 > E}$

(analog in a scalar field: $T_\mu{}^\mu$ can grow arbitrarily large if you feed back)

"Superradiance" & BH bombs

## Black Hole mechanics:

Laws that black holes are believed to satisfy.

Argument from Bekenstein that the only quantity of a black hole that changes is its area.

You may ask here! Why not the volume? This is because (related to the density argument) the volume is ill defined in the interior. This because volume as in geometry requires the choice of a codim 1 hypersurface that is unique. But this does not happen for a black hole and different "infalling" geometers will measure different volumes.

$$A = 4\pi r_s^2$$

using extremal surfaces

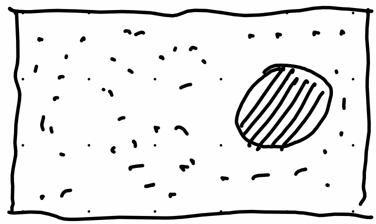
$$V \sim 3\sqrt{3}\,\pi M^2 v(t)$$

$v(t)$ is unbounded from above

So a black hole is not just a TARDIS, it's a TARDIS that behaves differently for different people.

So, the only increasing quantity measurable is the $r_s$ and area.

Jacob Bekenstein :



$$\frac{dS_{gas}}{dt} < 0 \longrightarrow \frac{dS_{BH}}{dt} > 0$$

But $\frac{\partial S}{\partial t}$ is measurable!

$$\frac{C_V}{T} = \frac{dS}{dT} = \frac{dS}{dt}\left(\frac{dT}{dt}\right)^{-1}$$

all measurable

The only other quantity is $\frac{dA}{dt}$, which $\geq 0$.

# The Hawking - Bardeen - Carter Laws

| | BH | Thermodynamics |
|---|---|---|
| 0th Law: | $\kappa$ is constant on $\mathcal{H}^+$ (only way to define $\kappa$ is on a bifurcate killing surface) | Systems in thermal eq. are at same temp. (only way to define temp). |

## ADM formalism:

We defined the Einstein field equations on $\Sigma_t$. We did not ask how to define "conserved energy".

This cannot be done the same way as in Gauss law and conserved charges.

When is energy conserved? $\partial_t$ is a killing symmetry.

"Globally": if $\partial_t$ is a global killing vector.

i.e if the spacetime admits a time like killing vector field. Can we define total energy in asy. flat spacetimes as a surface integral? As long as $\partial_t$ is a killing vector, yes.

$$g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu}$$ [ weak field approximation ] at $r \to \infty$

Pauli-Fierz identity: → (a key application: Massive gravity) & MOG IDE

$$\boxed{\begin{gathered}\underbrace{\eta^{\alpha\beta}\partial_\alpha\partial_\beta}_{\Box} h_{\mu\nu} + \eta^{\alpha\beta} h_{\alpha\beta,\mu\nu} - 2 h_{(\mu,\nu)} = \\ -2K\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T_{\alpha\beta}\eta^{\alpha\beta}\right) \\ K := 8\pi G.\end{gathered}}$$

Taking the trace:

$$\Box h + h^{\mu}{}_{,\mu} - \left(2h^{\mu}{}_{,\mu}\right) = -2K\left(T - 2T\right)$$

$$= \boxed{\Box h - 2h^{\mu}{}_{,\mu} = 2KT}$$

Typo in Townsend?

Consider: weak static dust as source

$$T_{\mu\nu} = \left(\begin{array}{c|c} \rho & 0_{3\times1} \\ \hline 0_{1\times3} & 0_{3\times3} \end{array}\right)$$ (pressure = 0 for dust)

$\dot{\rho} = 0 \Rightarrow$ static

$$\left.\begin{array}{l} 4\pi G\rho \ll 1 \\ T_{0i} = 0 \end{array}\right\} \Rightarrow \text{weak}$$

Static source $\Rightarrow$ $\dot{h}_{\mu\nu} = 0$ (assume)

$$\Box h_{00} + \cancelto{0}{h_{,00}} - 2\cancelto{0}{h_{(0,0)}} = -16\pi G\left(\rho - \frac{1}{2}\rho\right)$$

P-F eq $\leftarrow$ $\boxed{\nabla^2 h_{00} = -8\pi G\rho}$

Trace P-F:

$$\boxed{-\nabla^2 h_{00} + \nabla^2 h_{ii} - \underbrace{h_{ij,ij}}_{\partial_i(\partial_i h_{jj} - \partial_j h_{jj})} = -8\pi G\rho}$$

Add the two eq:

$$-16\pi G \rho = \partial_i (\partial_j h_{ij} - \partial_i h_{jj})$$

Treat $h_{ij}$ to be the metric on an almost flat surface at $\infty$.

$$E = \int_{\substack{t=\text{const} \\ \Sigma_{1\infty}}} d^3x \, T_{00}$$

$$= \frac{1}{16\pi G} \oint_{\Sigma_{1\infty}} \underbrace{dS_i}_{\substack{\text{Spacelike one-forms} \\ \text{"tangent to } \Sigma_1\text{"}}} (\partial_j h_{ij} - \partial_i h_{jj})$$

Mass / energy in GR are asymptotically defined quantities!

<u>Komar integrals</u> : Let $V \subset \Sigma_1$, $\partial V$ = Bdy of V

To every Killing vector field $\xi$, associate a Komar integral:

$$Q_\xi(V) = \frac{\text{const}}{16\pi G} \oint_{\partial V} \overbrace{dS_{\mu\nu}}^{\text{area element on } \Sigma_1 \text{ of } V} D^\mu \xi^\nu$$

You can conserve all quantities that have a killing vector conjugate.

Using Gauss' Law:

$$Q_\xi(V) = \frac{\text{Const}}{8\pi G} \int_V dS_\mu D_\nu D^\mu \xi^\nu$$

But for a killing vector field

$$\boxed{D_\mu D_\nu \xi^\nu = R_{\mu\nu} \xi^\nu}$$

$$Q_\xi(V) = \frac{\text{Const}}{8\pi G} \int_V dS_\mu R^\mu{}_\nu \xi^\nu$$

Einstein eq:

$$= \frac{\text{const}}{8\pi G} \int_V dS_\mu \left( T^\mu{}_\nu \xi^\nu - \frac{1}{2} T \xi^\mu \right)$$

$$= \int dS_\mu J^\mu(\xi)$$

Easy to check: $D_\mu J^\mu = 0$

(use $D_\mu T^{\mu\nu} = 0$)

if $\xi$ = time like Killing vector field

$$Q_K(V) = E(V) = \oint_{\partial V} dS_{\mu\nu}\, D^{\mu} K^{\nu}$$

$$\boxed{\text{Ex: if For } V \text{ such that } BH \subset V, \quad \partial V > 2M, \quad E(V) = M}$$

Similarly if $\xi = \partial_\phi$,

then $Q_{\partial_\phi}(V) = J(V)$ = angular mom.

---

Theorem: (Schoen - Yau - Witten):

ADM energy of asy-flat spacetime $(M, g)$, such that $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, is positive semi-definite.

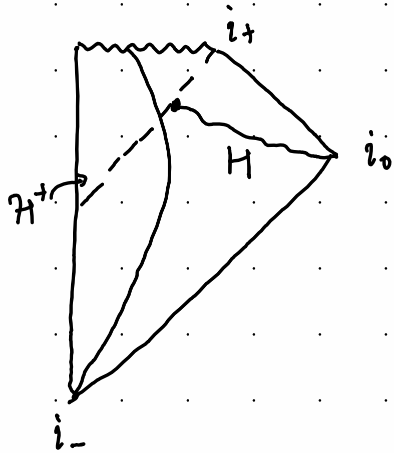
ADM energy of asy-flat spacetime is zero for only Minkowski, provided initial data is non-singular & $T_{\mu\nu}$ satisfies Dominant E.C.

## Remark:

1. Witten's Fields medal using supergravity
2. Instabilities (and QFT)
3. AdS $\longrightarrow$ negative $m^2$ is possible (Breitenlohner-Freedman bound)

## Return to BH mechanics:

Let $\Sigma$ be a spacelike hypersurface with inner boundary on $\mathcal{H}^+$ and outer bdy at $i_0$.




$H$ encompasses the black hole.

Apply Gauss Law to Komar integral of angular momentum $J$.

$$m = \partial_\phi$$

$$J = \frac{1}{8\pi G} \int_\Sigma dS_\mu D^\mu D_\nu m^\nu + \underbrace{\frac{1}{16\pi G} \oint_H dS_{\mu\nu} D^\mu m^\nu}_{J_H}$$

$$= \frac{1}{8\pi G} \int_\Sigma dS_\mu R^\mu{}_\nu m^\nu + J_H$$

$$\boxed{J = \int_\Sigma dS_\mu \left( T^\mu{}_\nu m^\nu - \frac{1}{2} T m^\mu \right) + J_H}$$

can be zero for an isolated black hole

Do the same for $n = \partial_t$. } Homework

$$\boxed{M = 2\Omega_H J - \frac{1}{8\pi G} \oint_H dS_{\mu\nu} D^\mu \xi}$$

where $\Omega_H$ is a constant on $H$.

But: $dS_{\mu\nu} = \xi_{[\mu}, n_{\nu]} \; dA$ on $H$

Where $\xi$ is a normal Killing vector and $n$ is such that $\xi \cdot n = -1$.

i.e $\xi$, $n$ are normal to $H$.

$$\therefore \frac{-1}{8\pi G} \oint_H dS_{\mu\nu} D^\mu \xi^\nu = \frac{-1}{4\pi G} \oint_H dA \, \underbrace{(\xi \cdot D\xi)^\nu}_{K\xi^\mu} n_\nu$$

$K$ ↓ surface gravity

(Measure of change of a $\partial_t$ quantity.)

$$\Rightarrow \quad = \frac{-K}{4\pi G} \oint dA \, \underbrace{\xi \cdot n}_{-1} = \frac{K A}{4\pi G}$$

$$\Rightarrow M = \frac{K A}{4\pi} + \Omega_H J \quad \text{(Kerr BH)}$$

(extremal: $K = 0$, $J = M$)
$\Omega_H = 1$.

Kerr Newman : $M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Phi_H Q$

co-rotating electic potential.

First Law of BH mechanics:

$$\boxed{dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ}$$

But $\kappa \Rightarrow T = \frac{\kappa}{2\pi}$

$$dM = \frac{T}{4} dA + \Omega_H dJ + \Phi_H dQ$$

Compare w/ First Law of thermodynamics

$$dE = T dS$$

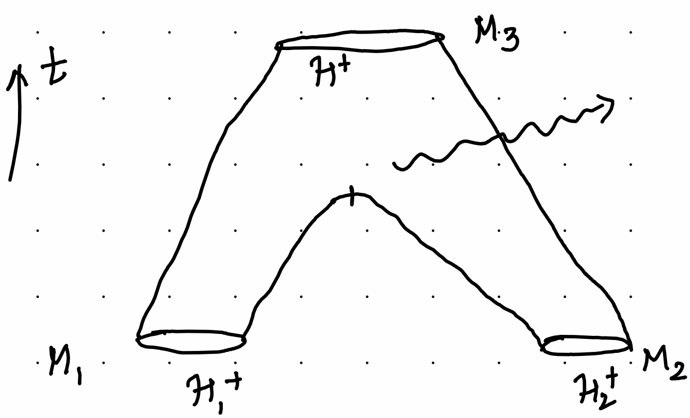
$$\Rightarrow dS = \frac{dA}{4} \Rightarrow \boxed{S = \frac{A_{BH}}{4}} \; !$$

Second Law of BH thermodynamics :

if $T_{\mu\nu}$ satisfies weak EC, asy flat $(M, g)$, no naked singularities exist, then

$$\frac{dA_{\mathcal{H}^+}}{dt} \geq 0$$

This limits the efficiency of BH's as sources of energy :




$$\Delta M = M_1 + M_2 - M_3$$

$$\eta = \frac{\Delta M}{M_1 + M_2} = 1 - \frac{M_3}{M_1 + M_2}$$

$$A_{H_1} = 4\pi(2M_1)^2 = 16M_1^2\pi$$

$$A_{H_2} = 16\pi M_2^2$$

Area Law: $A_3 \geq 16\pi(M_1^2 + M_2^2)$

But $16\pi M_3^2 \geq A_3$ (after ring down, equality)

$$M_3 \geq (M_1^2 + M_2^2)^{1/2}$$

$$\Rightarrow \eta = 1 - \frac{M_3}{M_1 + M_2} \leq 1 - \frac{(M_2^2 + M_1^2)^{1/2}}{M_1 + M_2}$$

$M_1 = M_2$ maximizer

$$\eta \leq 1 - \frac{\sqrt{2}}{2} \Rightarrow \eta \leq 1 - \frac{1}{\sqrt{2}}$$

$$\boxed{\eta \leq .3}$$

A BH is at most 30% efficient.

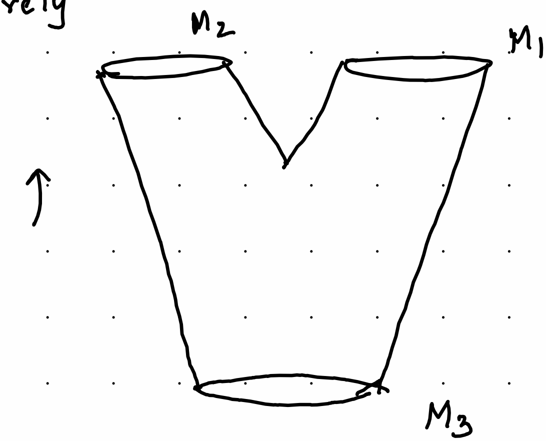
2. Black holes cannot "bifurcate"

$M_3 \longrightarrow M_1 + M_2$

Area Law : $M_3 \leq \sqrt{M_1^2 + M_2^2} \leq M_1 + M_2$

Energy cons : $M_3 \geq M_1 + M_2$ contradiction

Alternatively




Inverse process is at least 30% efficient w/ no upper bound. So this violates the Second Law.

Third Law : $K = 0$ can never be achieved in finitely many steps.

You will never be able to get a BH to have $M = J$ ...

# VIII : Black hole evaporation, Hawking radiation, & the information paradox

## 1. Quantum field theory (in curved spacetime)

Before we discuss curved space, let's do flat space.

$$(\Box + m^2)\,\phi(x) = 0 \quad , \quad \Box = \partial_\mu \partial_\nu \eta^{\mu\nu}$$

$\hookrightarrow$ Eigenmode decomposition:

You can decompose the periodic solutions into positive and negative modes

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega_k} \left( a_k e^{i(kx - \omega_k t)} + a_k^* e^{-i(kx - \omega_k t)} \right)$$

$$\omega_k = (k^2 + m^2)^{1/2}$$

in QFT: classical fields $\longrightarrow$ operators acting on a quantum field.

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega_k} \left( \hat{a}_k e^{i(k\hat{x} - \omega t)} + \hat{a}_k^{+} e^{-i(k\hat{x} - \omega t)} \right)$$

CCR: $\left[\hat{\phi}(x), \dot{\hat{\phi}}(y)\right] = i\delta^3(x-y)$

starting from a unique vacuum, you can generate a Fock space from $a, a^+$.

Let $(M, g)$ be a globally hyperbolic spacetime.

3+1 split:

$$ds^2 = -N\,dt^2 + h_{ij}\,(dx^i + N^i dt)(dx^j + N^j dt)$$

Let $\Sigma_t$ be a Cauchy surface of constant $t$.

Metric on $\Sigma_t := h$, $\sqrt{-g} = N\sqrt{h}$

$$(\Box + m^2)\,\phi(x) = 0$$

$$\Box = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\right)$$

Canonical conjugate momentum

$$\Pi(x) = -\sqrt{-g}\, g^{t\mu} \partial_\mu \Phi$$

$$= -N\sqrt{h}\,(dt)_\nu\, g^{\nu\mu} \partial_\mu \phi$$

$$= \sqrt{h}\, n^\mu \partial_\mu \phi$$

Quantization: "promote to operators"

$$[\hat{\phi}(x), \hat{\Pi}(x')](t) = i\delta^3(x-x')$$

$$[\hat{\phi}(x), \hat{\phi}(x')](t) = 0$$

$$[\hat{\Pi}(x), \hat{\Pi}(x')](t) = 0$$

What is the Hilbert space of states on which $\hat{\phi}$, $\hat{\Pi}$ act?

If $S$ is the set of all complex solutions to the KG equation, global hyperbolicity $\Rightarrow$ the set $S$ is specified by $(\Sigma, \phi, \partial_t \phi)$ and Cauchy data.

if $\alpha, \beta \in S$ then

$$(\alpha, \beta) = -\int_{\Sigma_0} d^3x \sqrt{h}\, n_a j^a(\alpha, \beta)$$

$$j(\alpha, \beta) = -i\,(\bar{\alpha}\, d\beta - \beta\, d\bar{\alpha})$$

$$\nabla^a j_a = -i\,(\bar{\alpha}\, \nabla^2 \beta - \beta \nabla^2 \bar{\alpha})$$

$$= -i\, m^2 (\bar{\alpha}\beta - \beta\bar{\alpha}) = 0 \Rightarrow j \text{ is conserved}$$

$\Rightarrow$ $\Sigma_0$ can be replaced by any $\Sigma_t$.

Note: • $(\alpha, \beta) = \overline{(\beta, \alpha)}$ i.e $(\cdot, \cdot)$ = Hermitian

• non-degenerate i.e if $(\alpha, \beta) = 0\ \forall\, \alpha \in S$ then $\beta = 0$.

But $(\alpha, \beta) = -(\bar{\beta}, \bar{\alpha}) \Rightarrow$

$(\alpha, \alpha) = -(\bar{\alpha}, \bar{\alpha}) \Rightarrow (\,,\,)$ is not positive def.

On Minkowski, $(\,,\,)$ is positive on +ve frequency subspace i.e in a spacetime with $K = \partial_t$,

$$\mathcal{L}_K \psi_p(x) = -ip^0 \psi_p(x)$$

$$\left[ S = S_p \oplus \bar{S}_p \right]$$

In curved spacetime: this notion of positive frequency does not hold in definition. [Blue shift....] In simple words, there is no preferred coordinate in curved space so there are many ways to choose $( , )$ to be positive definite. Meaning there is no preferred choice of $S_p$. Not all choices are equivalent Not all vacua are identical.

<u>Bogoliubov transform:</u> Let $\{\psi_i, \bar{\psi}_i\}$ form a basis of $S$. Let them satisfy the quantization algebra

$$\phi = \sum_j (c_j \psi_j + d_j \bar{\psi}_j)$$

$$= \sum_j (\hat{a}_j \psi_j + \hat{a}_j^{+} \bar{\psi}_j)$$

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = 0$$

Let $\{\psi_i', \bar{\psi}_i'\}$ be a different basis:
with creation / annihilation operators
$\hat{b}_i, \hat{b}_i^\dagger$.

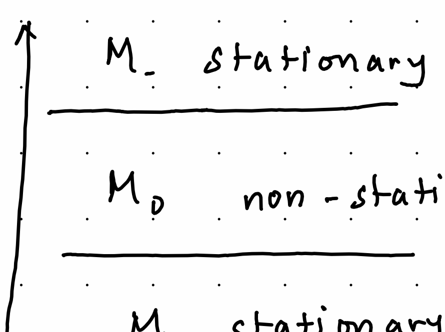
$$\psi_i' = \sum_j A_{ij}\psi_j + B_{ij}\bar{\psi}_j$$

$$\bar{\psi}_i' = \sum_j \bar{B}_{ij}\psi_j + \bar{A}_{ij}\bar{\psi}_j$$

"Bogoliubov transformations"

$A, B \rightarrow$ Bogoliubov coefficients

For $(M, g)$ globally hyperbolic:



In $M_\pm$, $\exists$ a choice of $S^\pm_p$ and notion of particles is well defined.

Let $\{u^\pm_i\}$ be an orthonormal basis for $S^\pm_p$. Let $\hat{a}^{\pm(\dagger)}_i$ be the operators...

$$u^+_i = \sum_j \left( A_{ij} u^-_j + B_{ij} \bar{u}^-_j \right)$$

$$a^+_i = \sum_j \left( \bar{A}_{ij} a^-_j - \bar{B}_{ij} a^{-\dagger}_j \right)$$

Vacua: $|0_\pm\rangle \Rightarrow a^\pm_i |0_\pm\rangle = 0$

Let $|0_-\rangle$ be early vacuum state. At late time,

\# of particles

$$\langle 0_- | N^+_i | 0_- \rangle = \langle 0_- | a^{+\dagger}_i a^+_i | 0_- \rangle$$

$$= \sum_{j,k} \langle 0_- | a^-_k (-B_{jk}) (-\bar{B}_{ij}) a^{-\dagger}_j | 0_- \rangle$$

$$= \sum_{jk} B_{jk} \bar{B}_{ij} \langle 0_- | a^-_k a^{-\dagger}_j | 0_- \rangle$$

$$= \sum_j B_{ij} \bar{B}_{ij} = (BB^\dagger)_{ii}$$

if $B = 0$ [i.e $S_p^+ = S_p^-$] then

$$\langle 0_- | N_i^+ | 0 \rangle = 0.$$

This happens if $M_+$ and $M_-$ are the same and no time dependent gravitational field has been turned on. if we insert $M_0$, then the gravitational field will generate particles.

This is the Hawking effect.

## Hawking radiation;

Let $M =$ schwarzschild

let $\phi$ be a KG field

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{r} \phi_{lm}(t, r) Y_{lm}(\theta, \phi)$$

$$\nabla^a \nabla_a \phi \Rightarrow \left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} - V_l(r_*) \right] \phi_{lm} = 0$$

$$V_l(r_*) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)$$

"Black hole potential": vanishes @ $r = 2M$
& @ $r = \infty$

Consider a solution describing a wave packet @ $r_*$ at $t_0$. $\lim t \to \infty$,
we can decompose the solution into two wave packets: $r_* \to -\infty$ $(r = 2M)$
and $r_* \to \infty$ $(r = \infty)$

At early time $t \to -\infty$:

$$\phi_{lm} = f_\pm(u) + g_\pm(v)$$ i.e

superposition of left / right wave packets.

@ late time: $t \to \infty$

$$f_+(u) \to \mathscr{I}^- \quad \text{"outgoing"}$$

$$g_+(u) \to \mathcal{H}^+ \quad \text{"ingoing"}$$

Solution = superposition of solutions that vanish on $\mathcal{H}^+$ and $\mathcal{I}^+$.

Similarly : $f_-(u)$, $g_-(u)$ $\rightarrow$ inward from $\mathcal{I}^-$

$\downarrow$

outward from $\mathcal{H}^-$

Since spacetime is stationary :

$$\phi_{\omega l m} = \frac{1}{r} e^{-i\omega t} R_{\omega l m}(r) Y_{lm}(\theta, \phi)$$

$\omega > 0$

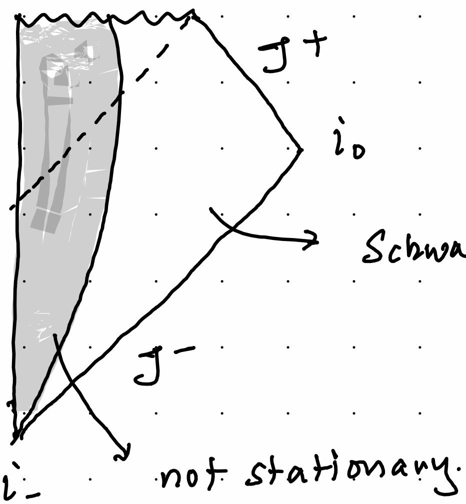
Let $\phi_{lm} = e^{-i\omega t} R_{\omega l m}$

$$\Rightarrow \left[ -\frac{d^2}{dr_*^2} + V_l(r_*) \right] R_{\omega l m} = \omega^2 R_{\omega l m}$$

"Schrödinger like equation"

Hawking radiation:

Consider a massless scalar field of particles describing a shell collapse




So expect particle creation.

@ $t = -\infty$, no $\mathcal{H}^-$ so only ingoing modes

@ $t = \infty$, ingoing ($\mathcal{H}^+$) and outgoing ($\mathcal{J}^+$) modes

$\mathcal{J}^-$, $\mathcal{J}^+$ modes are positive freq since spacetime is static!

Modes on $\mathcal{H}^+$ are not positive frequency because $\mathcal{H}^+$ is not static at BH formation times.

$\mathcal{I}^-$ : $\{f_i, \bar{f}_i\}$ : $(f_i, f_j) = \delta_{ij}$

$\mathcal{I}^+$ : $\{p_i, \bar{p}_i\}$ $(p_i, p_j) = \delta_{ij}$

$\mathcal{H}^+$ : $\{q_i, \bar{q}_i\}$ $(q_i, q_j) = \delta_{ij}$

$f_i$ : $\hat{a}_i, \hat{a}_i^+$

$p_i$ : $\hat{b}_i, \hat{b}_i^+$

$q_i$ : $\hat{c}_i$ ; $\hat{c}_i^+$

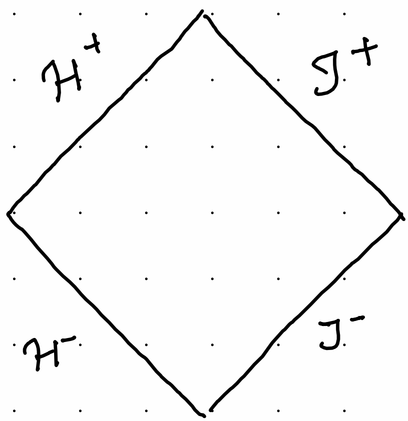
Expand in Bogoliubov basis:

$$p_i = \sum_j \left( A_{ij} f_j + B_{ij} \bar{f}_j \right)$$

$$\Rightarrow \quad b_i = \sum_j \left( \bar{A}_{ij} a - B_{ij} a_j^+ \right)$$

At early times, we assume we are in a vacuum. $a_i|0\rangle = 0$

To calculate # of outgoing modes at $\mathscr{I}^+$, we need $B_{ij}$ coefficients




Consider a wave solution traced backward from $\mathcal{H}^+ \cup \mathscr{I}^+$

Part of this wave is reflected in to $\mathscr{I}^-$, part is transmitted to $\mathcal{H}^-$.

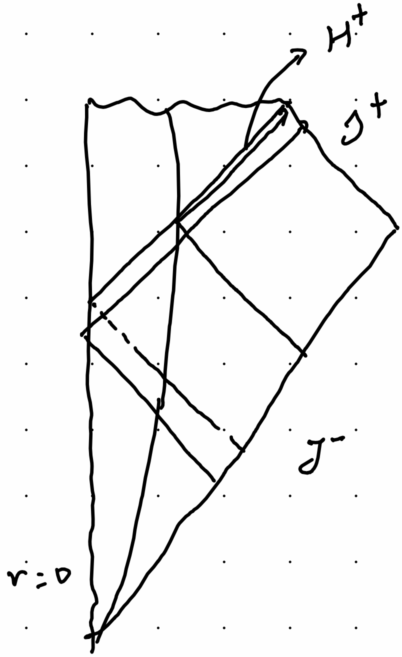
$$\overset{\mathcal{H}^+ \cup \mathscr{I}^+}{p_i} = \overset{\to \mathscr{I}^-}{p_i^{(1)}} + p_i^{(2)} \to \mathcal{H}^-$$

$$R_i = \sqrt{(p_i^{(1)}, p_i^{(1)})}, \quad T_i = \sqrt{(p_i^{(2)}, p_i^{(2)})}$$

Normalization: $R_i^2 + T_i^2 = 1$

Time reversal: $T_i^2$ is the fraction of waves from $\mathscr{I}^-$ to cross $\mathcal{H}^+$, $R_i^2$ is the fraction that gets reflected into $\mathscr{I}^+$.

Now, if we include collapsing matter,




$P_i^{(1)} \longrightarrow$ Reflected to $\mathscr{I}^+$

$P_i^{(2)} \longrightarrow \mathcal{H}^+$

Time reversal: the modes from $\mathscr{I}^+$ when extended backward cross the $\underbrace{\text{collapsing shell}}$ and go to $\mathscr{I}^-$.

→ A time dependent geometry

The wave that reflect into the shell

are now decomposed in +ve and -ve modes

$\therefore$ $P_i^{(2)}$ determines $B_{ij}$

$$A_{ij} = A_{ij}^{(1)} + A_{ij}^{(2)}$$

$$B_{ij} = B_{ij}^{(2)}.$$

Q: How do we relate $A_{ij}$ to $B_{ij}$?

I don't present the calculation here as it is a pretty heavy QFT calculation. Let me state the result.

$$|B_{ij}| = e^{-w_i \pi / K} \, |A_{ij}^{(2)}|$$

$$T^2 = \left(P_i^{(2)}, P_i^{(2)}\right) = \sum_j \left( |A_{ij}^{(2)}|^2 - |B_{ij}|^2 \right)$$

$$= \left(e^{2\pi w_i / K} - 1\right) \sum_j |B_{ij}|^2$$

$$= \left(e^{2\pi w_i / K} - 1\right) (BB^+)_{ii}$$

$$\langle 0 | b_i^+ b_i | 0 \rangle$$

$$\langle 0 | b_i^\dagger b_i | 0 \rangle = \frac{T^2}{\left(e^{2w_i \pi / \kappa} - 1\right)}$$

i.e the spectrum of particles created from quantum effects of a BH is thermal with temperature $T_H = \frac{\kappa}{2\pi}$.

(Black holes are black bodies)

We can solve this for any field. (Say Dirac) Then, that is where we see that black holes prefer to emit particles of the same charge.

- $T_H$ is very small for astrophysical / SMBH's. Primordial tests?
- No preventive mechanism to stop this emission. So BH's evaporate
- Breakdown of predictability

$$M_- \cup M_+ \to M_- \cup M_0 \cup M_+ \to M_- \cup M_+$$

- $T_H$ decreases with increasing $M$
  $\Rightarrow$ Larger the BH, the lower its heat capacity

- Modification of $2^{nd}$ Law:

$$dS = \frac{dA}{dt} \leq 0$$

$$dS = dA + S_{matter}$$

HW: (1) Entropy of a solar mass BH : $10^{77}$
Entropy of the sun : $10^{58}$

Black holes form eventually in the dS universe. "BH dominated phase"
$\sim 10^{43}$ years

<u>Evaporation</u>: Since a BH behaves like a black body,

$$\frac{dE}{dt} = -\alpha \underbrace{T^4 A}_{\sim T^2} \quad \text{approximations}$$

$$A \propto M^2, \; T \propto \frac{1}{M} \Rightarrow \frac{dE}{dt} \propto -\frac{1}{M^2}$$

$$T_{Evap} \simeq 10^{71} \left(\frac{M}{M_\odot}\right)^3 \text{ seconds}$$

$\downarrow$

Boundary condition is that BH size is $\sim l_p$.

i.e it is a classical result.