The Kerr and Kerr-Newman spacetimes

Def: A spacetime asymptotically flat at null infinity (i.e asymptotically Minkowski) is stationary if it admits a killing vector field that is timelike in a neighbourhood of $\mathcal{J}^{\pm}$. if K^a is hypersurface orthogonal, it is static.

Def: if K^a above is spacelike near $\mathcal{J}^{\pm}$ and if it generates a 1 parameter group of isometries isomorphic to $U(1)$, M is axisymmetric.

Theorem (Israel): if (M, g) is a static BH spacetime (asy flat), M is schwarzschild upto isometries.

Theorem (Hawking; Wald): if (M, g) is a stationary, non-static, asy flat solution of Einstein-maxwell action, then (M, g) is axisymmetric.

Theorem (Carter): if (M, g) is a stationary axisymmetric, asy flat vacuum solution outside a BH, then (M, g) is a 2-parameter family of solution; params: M, J.

Generically: BH's in the universe are Kerr.

Extendible to 4 param (M, Q, P, J)

"Kerr-Newman:

Derivation ?

Kerr-Newman solution: (M, Q, P, J)

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma_1}\right) dt^2 - \frac{2a\sin^2\theta\,(r^2+a^2-\Delta)}{\Sigma_1} *$$

$$* \, dt\, d\phi + \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma_1}\right)\sin^2\theta\, d\phi^2$$

$$+ \frac{\Sigma_1}{\Delta} dr^2 + \Sigma_1 d\theta^2$$

"Boyer-Lindquist" coords

$$A = -\frac{Qr}{\Sigma_1}(dt - a\sin^2\theta\, d\phi) + \frac{P\cos\theta}{\Sigma_1} *$$

$$* \, (a\,dt - (r^2+a^2)\,d\phi)$$

$$\Sigma_1 = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2 + e^2$$

$$e = \sqrt{Q^2 + P^2}$$

$$\lim_{a \to 0} (ds^2, A)_{KN} \longrightarrow (ds^2, A)_{RN} \quad \checkmark$$

$$\lim_{\substack{a \to 0 \\ P, Q \to 0}} (KN) \longrightarrow \text{Schwarzschild.} \quad \checkmark$$

$$\lim_{P, Q \to 0} (KN) \longrightarrow \text{Kerr.}$$

Kerr Solution :

Set $e = 0$ in the Kerr - Newman metric

Kerr metric : in Boyer - Lindquist coords :

$$ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} dt^2 - \frac{2a \sin^2\theta (r^2 + a^2 - \Delta)}{\Sigma} dt\, d\phi$$

$$+ \frac{\left((r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right)}{\Sigma} \sin^2\theta\, d\phi^2$$

$$+ \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 ,$$

$$a := \frac{J}{M}$$

$$\Sigma = r^2 + a^2 \cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

$$\Delta = (r + r_-)(r + r_+)$$

$$r_\pm = M \pm (M^2 - a^2)$$

① There is no Birkhoff's theorem for Kerr spacetime.

② $g_{tt} = 0 \Rightarrow \Delta = 0, \theta = 0$

Singularity: $\Sigma = 0 \Rightarrow r = 0, \theta = \frac{\pi}{2}$

Consider:

$$r_{\pm} = M \pm (M^2 - a^2)^{1/2}$$

Case: $M^2 < a^2$

Δ has no real roots, but the metric still has singularities $\Rightarrow$ naked! (excluded)

To understand this better: go to Kerr-Schild coordinates:

$$x + iy := (r + ia) \sin\theta \, e^{i\int(d\phi + \frac{a}{\Delta} dr)}$$

$$z := r\cos\theta$$

$$\tilde{t} := \int\left(dt + \frac{r^2 + a^2}{\Delta} dr\right) - r$$

Plug in:

$$r^4 - (x^2+y^2+z^2-a^2)\, r^2 - a^2 z^2 = 0$$

$$ds^2 = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr^3}{r^4+a^2z^2}\left[\frac{r(x\,dx+y\,dy) - a(x\,dy - y\,dx)}{r^2+a^2} + \frac{z\,dz}{r} + d\tilde{t}\right]^2$$

See that $\lim M \to 0$ of above is Minkowski.

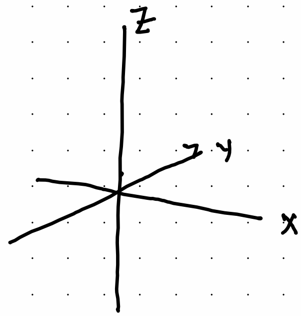
Solve for surface of const $\hat{t}, r$:

Let $r=0$ be the limit we take. ($z=0$)

ie colimit, $r, z \to 0$.

degenerates

$$x^2+y^2-a^2=0$$



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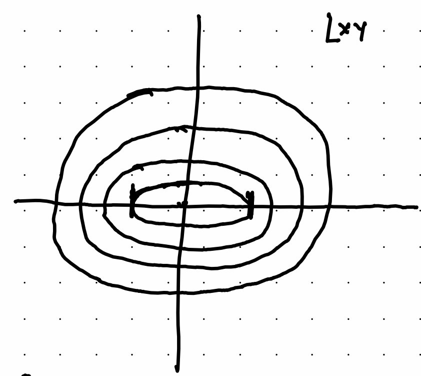
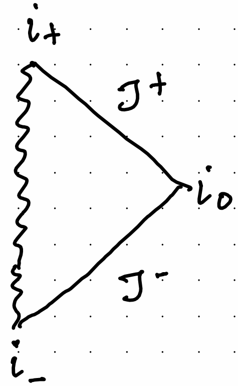


$$x^2+y^2=a^2$$

$\theta = \frac{\pi}{2} \rightarrow$ The singularity is the ring $x^2+y^2=a^2$, $z=0$

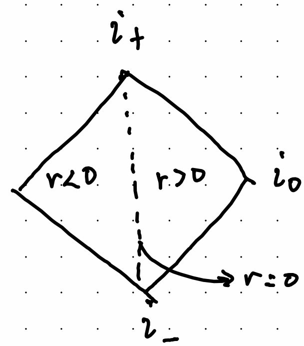
Causal structure of $M^2 < a^2$:

$\theta = \frac{\pi}{2}$



naked singularity at $r=0 \Rightarrow$ $x^2+y^2=a^2$

$\theta = 0$



if you start @ $r>0$, you can pass through the ring $x^2+y^2=a^2$ at $r=0$ into $r<0$.

Also

Notice: g is independent of ϕ.

$m = \partial_\phi$ is a Killing vector.

$$m^2 = g_{\phi\phi} = a^2\sin^2\theta\left(1+\frac{r^2}{a^2}\right) + \frac{M^2a^2}{r}\left(\frac{2\sin^4\theta}{1+\frac{a^2}{r^2}\cos^2\theta}\right)$$

Let $r/a = \delta$, $r<0$, $\delta \ll 1$

$\theta = \pi/2 + \delta$ [you are close to singularity]

$$m^2 \simeq a^2 + \frac{Ma}{\delta} + \theta(\delta).$$

$$m^2 \simeq a^2 - \frac{Ma}{|\delta|},$$ which $\Rightarrow m^2 < 0$ for small enough $|\delta|$.

So m^2 = timelike near ring singularity.

But motion in ϕ direction is periodic and $\therefore$ ∂_ϕ should have closed orbits meaning your spacetime has closed timelike orbits. $\downarrow$ CTC's

Global violation of causality.

Case $M^2 > a^2$:

The same ring singularity exists but r_+, r_- are singular radii ie horizons.

These are coordinate singularities:

$$dv := dt + \frac{(r^2 - a^2)}{\Delta} dr$$

$$d\chi := d\phi + \frac{a}{\Delta} dr$$

(Kerr coords)

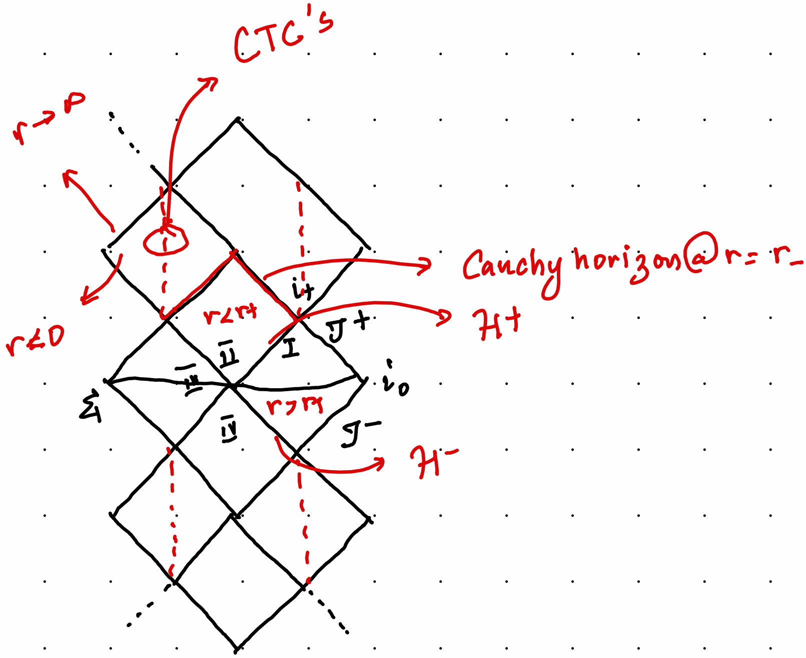
actually disovered first.

Analogous to IEF:

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} dv^2 + 2\,dv\,dr - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma} dv\,d\chi - 2a\sin^2\theta\,d\chi\,dr + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,d\chi^2 + \Sigma\,d\theta^2$$

The singularity at $\Delta = 0$ no longer persists.

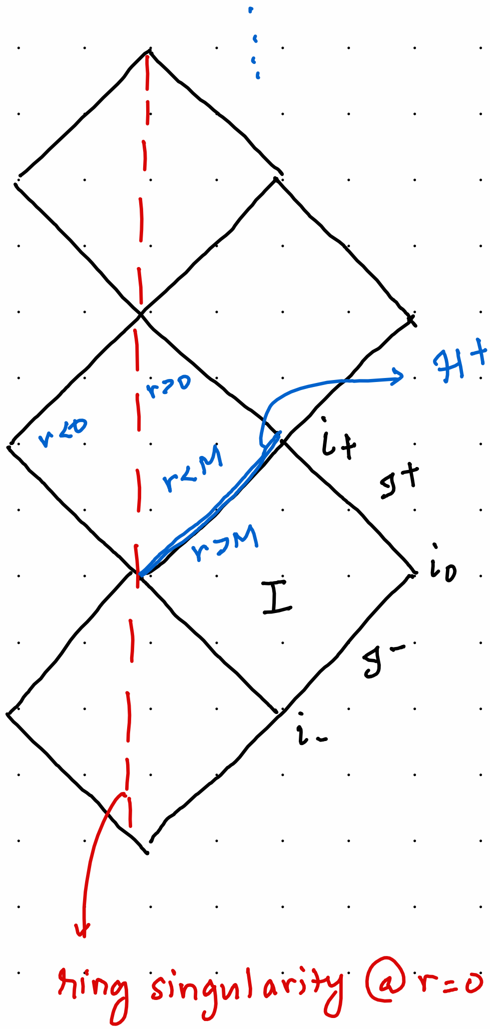
Analogous to RN: $\kappa_\pm = \dfrac{r_\pm - r_\mp}{2(r_\pm^2 + a^2)}$



Penrose - Carter diagram of Kerr.

<u>Case</u>: $M^2 = a^2$

$r_+ = r_-, \quad K_{\pm} = 0$



You could in principle live forever inside an extremal Kerr Black hole.

The Ergosphere:

∂_t is a Killing vector of Kerr
(stationary)

$$k^2 = g_{tt} = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} = -\left(1 - \frac{2Mr}{r^2+a^2\cos^2\theta}\right)$$

time like: $$1 - \frac{2Mr}{r^2+a^2\cos^2\theta} > 0$$

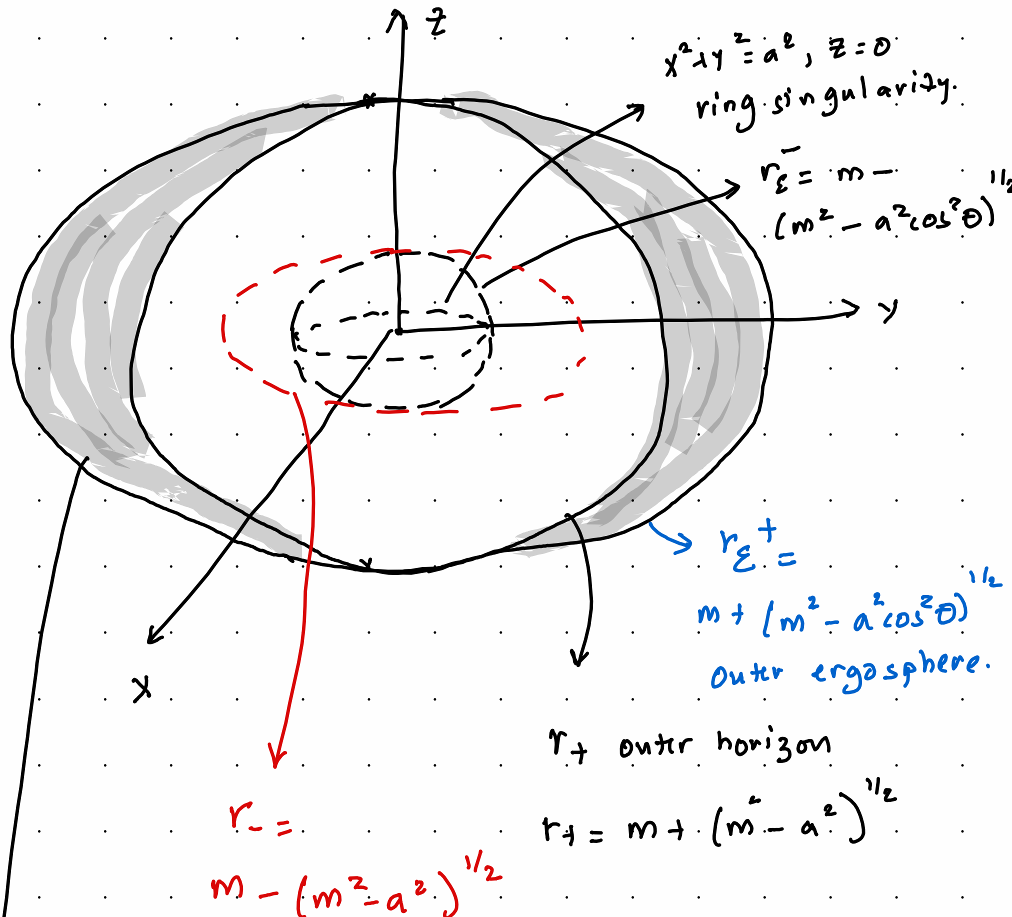
$$\Rightarrow \quad 2Mr < r^2 + a^2\cos^2\theta$$

$$\Rightarrow r^2 + a^2\cos^2\theta - 2Mr > 0$$

$$r_E = M \pm (M^2 - a^2\cos^2\theta)^{1/2}$$

is the hypersurface "ergosphere".

$\theta = 0, \pi$, $r_E = r_+$

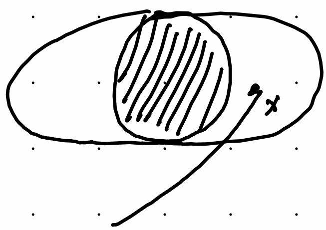


Shaded Region "Ergosphere"

Region where K becomes spacelike outside EH. Meaning that you cannot stay still!

You will "corotate" with the BH.

Black hole slingshots :



@ x : Fire a propellant in a "counter rotating" orbit.

The Body can now have enough energy to escape the ergoregion with ΔE,

$$\Delta E \leq \begin{cases} 0.21\, M_{Body}, & \text{Kerr.} \\ 0.29\, M_{Body}, & \text{Kerr Newman} \end{cases}$$

let 4-momentum of a body be p. It approaches a Kerr BH along a geodesic

$$E = -p \cdot K$$

@ x, $E_1 = -p_1 \cdot K$, $E_2 = -p_2 \cdot K$
(into BH)

$E_2 = E - E_1$

$= E + p \cdot K$. But if this happens inside the ergoregion, K is spacelike

$\therefore p_\mu K^\mu > 0$!

$\therefore \boxed{E_2 > E}$

(analog in a scalar field: $T_\mu{}^\mu$ can grow arbitrarily large if you feed back)

"Superradiance" & BH bombs