Penrose - Carter diagrams:

where is the ∞ in asymptotic infinity?
It is possible to "bring" this ∞ into spacetime by conformal compactification. (This does not change the causal structure of spacetime).

All points that are ∞ly far away in proper distance are only finitely far away in terms of the affine parameter of a new metric.

Example: $\mathrm{Mink}_{1,d-1}$, $d=4$

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$$

$$\begin{Bmatrix} u = t-r \\ v = t+r \end{Bmatrix} \longrightarrow ds^2 = -du\,dv + \frac{(u-v)^2}{4} d\Omega_2^2$$

set $\quad u = \tan U \quad , \quad V, U \in (-\pi/2, \pi/2)$

$\quad\quad v = \tan V \quad\quad\quad V > U.$

$$ds^2 = (2\cos U \cos V)^{-2} \left[-4\,dU\,dV + \sin^2(V-U)\,d\Omega_2^2\right]$$

if either/and $U, V = \frac{\pi}{2}$, we get $ds^2 = \infty$.

$\Lambda = 2\cos U \sin U.$

$$d\tilde{s}^2 = \Lambda^2 ds^2 = -4\,dU\,dV + \sin^2(V-U)\,d\Omega_2^2.$$

In this new conformally scaled metric, we can add points at ∞. $V \geq U$

$$\left.\begin{array}{l} U = -\frac{\pi}{2} \\ V = \frac{\pi}{2} \end{array}\right\} \longrightarrow \begin{array}{l} u = -\infty \\ v = \infty \end{array} \longrightarrow \begin{array}{l} r \to \infty \\ t \text{ finite} \end{array} : \begin{array}{l} \text{Spatial } \infty \\ i_0 \end{array}$$

$$\left.\begin{array}{l} U = \pm\frac{\pi}{2} \\ V = \pm\frac{\pi}{2} \end{array}\right] \longrightarrow \begin{array}{l} u = \pm\infty \\ v = \pm\infty \end{array} \longrightarrow \begin{array}{l} t \to \pm\infty \\ r \text{ finite.} \end{array} : \begin{array}{l} \text{past / future} \\ \text{temporal } \infty \\ i_\pm . \end{array}$$

$$\begin{array}{l} U = -\frac{\pi}{2} \\ |V| \neq \pi/2 \end{array} \longrightarrow \begin{array}{l} u = -\infty \\ v = \text{finite.} \end{array} \longrightarrow \begin{array}{l} r \to \infty \\ t \to -\infty \\ r+t \text{ finite} \end{array} \longrightarrow \begin{array}{l} \text{Past null } \infty . \\ \mathscr{J}^- \end{array}$$

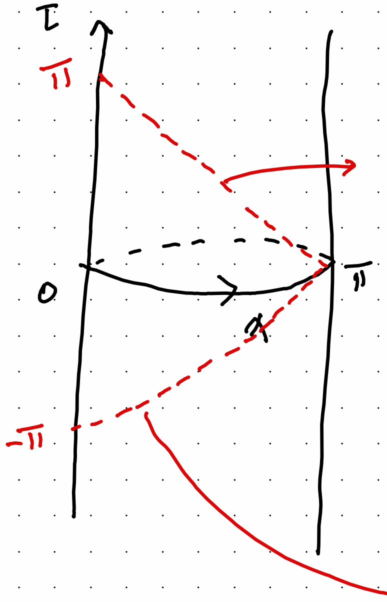
$$\begin{array}{l} |U| \neq \frac{\pi}{2} \\ \tilde{V} = \frac{\pi}{2} \end{array} \longrightarrow \begin{array}{l} u \text{ finite} \\ v = \infty \end{array} \longrightarrow \begin{array}{l} r \to \infty \\ t \to \infty \\ r-t \text{ finite} \end{array} \longrightarrow \begin{array}{l} \text{Future null } \infty \\ \mathscr{J}^+ \end{array}$$

$ds^2_{\text{mink}} \hookrightarrow d\tilde{s}^2$, boundary at $\Lambda = 0$

Let $\tau = V + U$, $\chi = V - U$

$$d\tilde{s}^2 = -d\tau^2 + d\chi^2 + \sin^2\chi \, d\Omega_2^2$$

$$\Lambda = \cos\tau + \cos\chi . \qquad \chi \sim \chi + 2\pi$$



if each of the 2 sphere at const χ is a point,

$\chi + \tau = \pi, \ v = \pi/2, \ \mathcal{J}^+$

Mink:

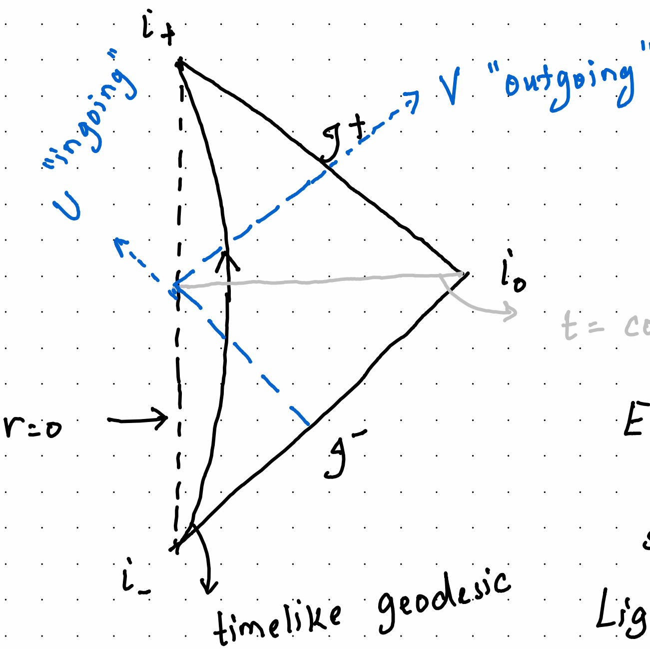
$-\pi < \tau < \pi$

$0 \leq \chi \leq \pi$

$-\pi < \chi + \tau \leq 2\pi$

$\chi - \tau = \pi, \ \tilde{u} = -\frac{\pi}{2}, \ \mathcal{J}^-$

Now, squash the cylinder.



Each point (except i_0, i_+, i_-) are 2-spheres.

Light rays / null vec ($\mathcal{J}^- \to \mathcal{J}^+$)

massive $\to$ $i_- \to i_+$.

Example: Kruskal spacetime:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv + r^2 d\Omega_2^2 \qquad (I)$$

$u = \tan U \quad , \quad U \in (-\pi/2, \pi/2)$

$v = \tan V \quad \quad V \in (-\pi/2, \pi/2)$

$$ds^2 = (2\cos V \cos U)^{-2} \left[-4\left(1 - \frac{2M}{r}\right) dU\, dV + r^2 \cos^2 U \cos^2 V \, d\Omega_2^2 \right]$$

$$2r^* = v - u \Rightarrow r^* = \frac{\tan V - \tan U}{2} = \frac{\sin(V-U)}{2\cos V \cos U}.$$

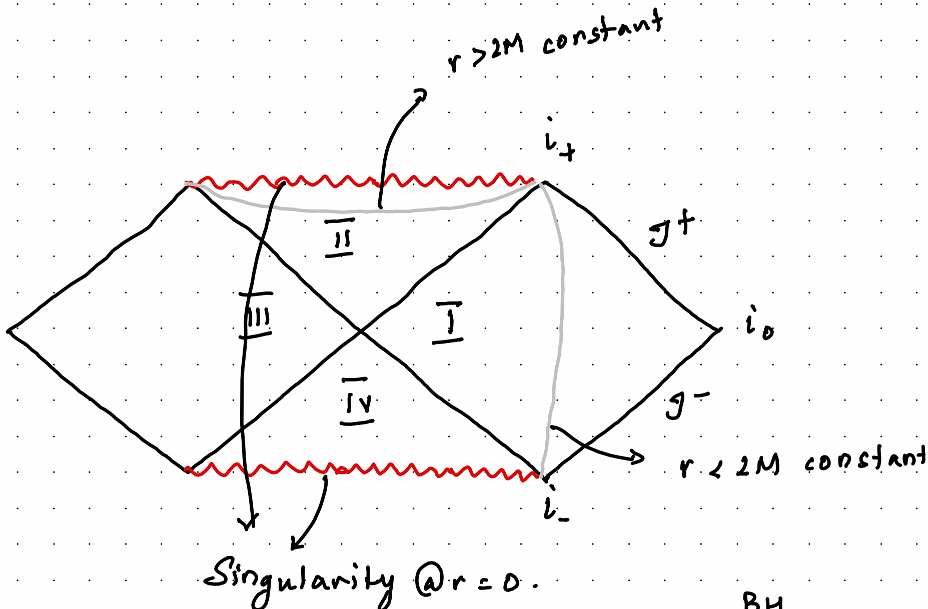
$$r^* = r + 2M \ln\left(\frac{r-2M}{2M}\right)$$

$$d\tilde{s}^2 = \Lambda^2 ds^2 = -4\left(1 - \frac{2M}{r}\right) dU\, dV + \left(\frac{r}{r_*}\right)^2 \sin^2(V-U)\, d\Omega_2^2$$

$$\lim_{r \to \infty} d\tilde{s}^2 = -4\, dU\, dV + \sin^2(V-U)\, d\Omega_2^2$$

which is ds^2_{Mink}.

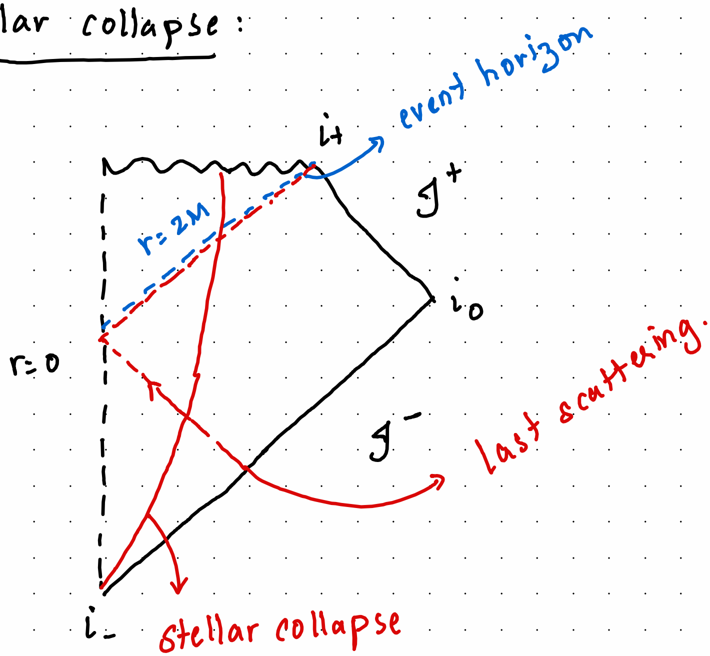
Kruskal is "asymptotically flat"! So we can add $\mathscr{I}^\pm$.



"Penrose diagram for a Kruskal BH spacetime"

All r = const hypersurfaces meet @ i_+.

Stellar collapse:



Let M be an asymptotically flat spacetime.

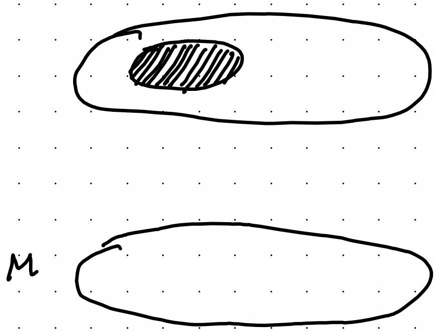
Let $U \subset M$. $J^-(U)$ is the causal past of U.

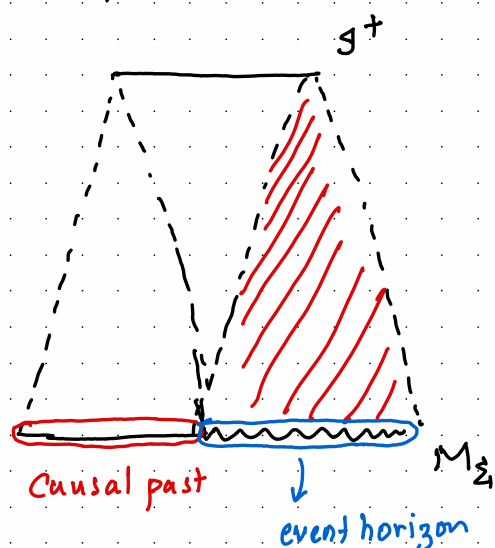
$\bar{J}^-(U)$ = closure of J^- including limit points
(Union of U and its Bdy)

$\partial \bar{J}^-(U) := \dot{J}^-(U) = \bar{J}^-(U) - J^-(U)$

Future event horizon of M is

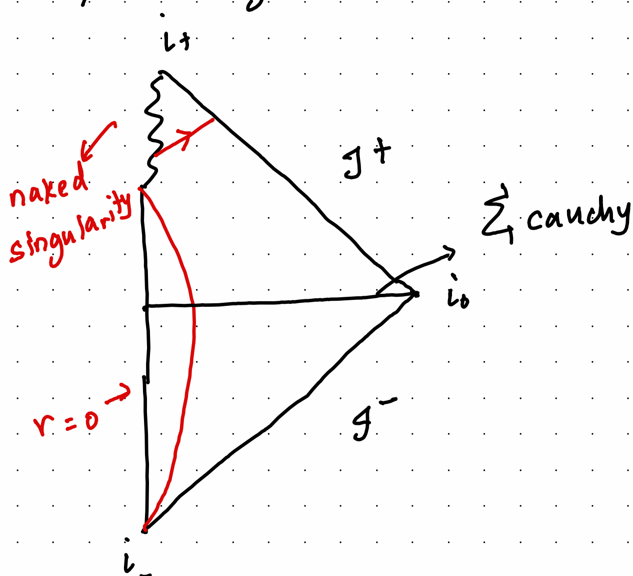
$\mathcal{H}^+ = \dot{J}^-(\mathscr{I}^+)$ i.e. bdy of closure of causal past of $\mathscr{I}^+$.





- $i_0, \mathscr{I}^- \notin \mathcal{H}^+$
- $\mathcal{H}^+$ is a hypersurface (null).
- No two points on $\mathcal{H}^+$ are timelike separated
- generators of $\mathcal{H}^+$ may have past end points
- generators of $\mathcal{H}^+$ have no future end points (singularity thm)

The past singularity in Kruskal is naked.



if this were possible, then $(h_{ab}, K_{ab}, \Sigma_1 \text{ cauchy})$ cannot predict (g_{ab}, G_{ab}) on $\mathcal{J}^+$.

Cosmic Censorship Conjecture:

Naked singularities cannot form in asy. flat M by grav collapse if $\Sigma_1(t)$ is non-singular for some t.

↳ Unsolved!

Reissner - Nordström black holes

These are black holes charged under a $U(1)_{EM}$ field.

$$S = \frac{1}{2K} \int d^4x \sqrt{-g} \left[R - F_{\mu\nu} F^{\mu\nu} \right]$$

Unusual normalization of the Maxwell term.

$$\left.\begin{aligned} G_{\mu\nu} &= 2\left(F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \\ D_\mu F^{\mu\nu} &= 0 \end{aligned}\right\} \text{source free}$$

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)} + r^2 d\Omega_2^2$$

actually $Q \sim Q(M)$

$Q(M=0) = 0$ (in fact we will see that this is relevant)

$A = \frac{Q}{r} dt$ ($F = dA$, 1- form Maxwell potential)

Birkhoff thm : generalization for EM equations

$$ds^2_{RN} = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\Omega_2^2$$

$$\Delta = r^2 - 2Mr + Q^2$$

$\Delta = 0 \longrightarrow r_{\pm} = M \pm \sqrt{M^2 - Q^2}$

$M < |Q|$, then we have naked singularities X

HW: Consider a collapsing shell of matter, mass M, charge Q. Consider total energy/mass as a function of R. Show that the collapsing shell is possible iff $M > Q$.

Sort of obvious:

$$\frac{GM^2}{R} = \frac{GQ^2}{R} \longrightarrow$$ Coulomb potential is holding the grav pot. in eq.

To analyze $M > |Q|$, we introduce the EF coords

For $r > r_+$: $dr_* = \frac{r^2}{\Delta} dr$

$$r_* = r + \frac{1}{2K_+} \ln \left| \frac{r - r_+}{r_+} \right| + \frac{1}{2K_+} \ln \left| \frac{r - r_-}{r_-} \right| + \text{const}$$

(check this)

where $K_{\pm} = \frac{(r_{\pm} - r_{\mp})}{2r_{\pm}^2}$

These are the inner and outer surface gravities i.e acceleration of a particle at the horizon, as measured by an observer at ∞

Now, $u = t - r_*$, $v = t + r_*$

(Ingoing EF coordinates)

$$ds^2 = -\frac{\Delta}{r^2} dv^2 + 2 dv dr + r^2 d\Omega_2^2$$

This metric is smooth for $r > 0$ and can therefore be continued to $r < r_+$.

r_+ : Outer event horizon
r_- : inner event horizon
$\}$ both null hypersurfaces

(analogously, using outgoing EF coordinates)

$$ds^2 = -\frac{\Delta}{r^2} du^2 - 2 du dr + r^2 d\Omega_2^2$$

(white hole of RN solution)

For the case: $M = |Q|$, the two horizons coincide $r_+ = r_-$ and $K_+ = K_- = 0$. This means that the RN solution is extremal and has no surface grav.

Also, $T = \dfrac{\kappa}{2\pi}$ (Temp $\propto$ surface gravity)

So extremal solutions have zero temperature.

Kruskal - coordinates for RN black holes:

We want to understand the global structure of the Reissner-Nordström solution. So we introduce Kruskal like coordinates:

$$U^{\pm} = -e^{-\kappa_{\pm} u} \quad , \quad V^{\pm} = \pm e^{\kappa_{\pm} v}$$

Start in $r > r_+$. (v^+, u^+, θ, ϕ)

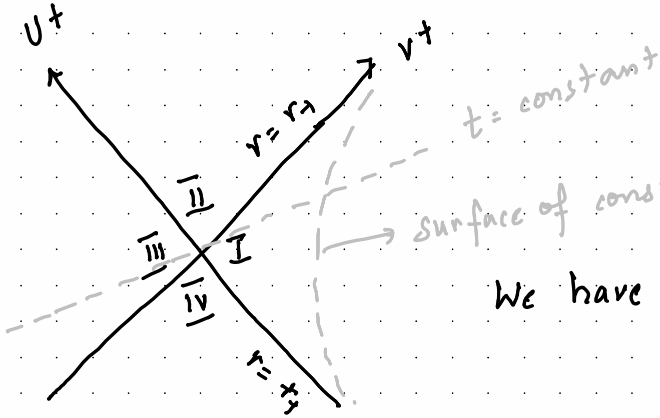
$$ds^2 = -\frac{\Delta}{r^2} dr^2 + 2\, dv\, dr + r^2 d\Omega_2^2$$

HW: $$ds^2 = -\frac{r_+ r_-}{\kappa_+^2 r^2} e^{-2\kappa_+ r} \left(\frac{r - r_-}{r_-}\right)^{1 + \frac{\kappa_+}{|\kappa_-|}} dU^+ dV^+$$

$$+ r^2 d\Omega_2^2$$

$r(u^+, v^+) \Rightarrow$

$$-U^+ V^+ = e^{2\kappa_+ r} \left(\frac{r - r_+}{r_+}\right) \left(\frac{r_-}{r - r_-}\right)^{\frac{\kappa_+}{|\kappa_-|}}$$



(ER bridge)

We have $U^+ < 0$, $V^+ > 0$
$\Rightarrow r > r_+$

No singularities in $\overline{\text{II}}$ an $\overline{\text{IV}}$ because $r(U^+, V^+) > r_-$. if we continue to $U^+ \geq 0$ and $V^+ \leq 0$, we will see singularities.

$U^+ = V^+ = 0$: Null hypersurfaces intersect at a bifurcation 2-sphere.

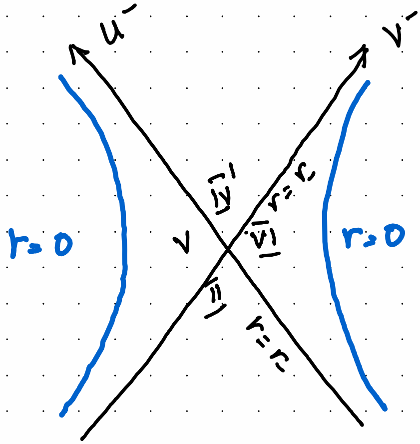
To investigate the RN metric at $r \leq r_-$,

define: $v = t + r_*$
the retarded coordinates $u = t - r_*$

and use U^-, V^- as before:

This now gives $U^-, V^- < 0$ in $\overline{\text{II}}$:

$$ds^2 = \frac{-r_+ r}{K_-^2 r^2} e^{2|K_-| r} \left(\frac{r_+ - r}{r_+}\right)^{1 + |K_-|/K_+} dU^- dV^- + r^2 d\Omega^2$$



$$U^-V^- = e^{-2|\kappa_-|r}\left(\frac{r-r_-}{r_-}\right) \times \left(\frac{r_+}{r_+-r}\right)^{|\kappa_-|/\kappa_+}$$

We have new regions $\overline{V}$ and $\overline{VI}$ $\quad 0 < r < r_-$ which contain curvature singularities at $r=0$ $(U^-V^-=-1)$. $\overline{IV}' \sim \overline{IV}$ by isometry and can be continued to new regions $\overline{I}', \overline{II}', \overline{III}'$.. $\overline{I}'$ and $\overline{III}'$ are new regions (asy. flat) isometric to $\overline{I}$ and $\overline{III}$.

RN solutions are fascinating:

Consider a path of constant r, θ, ϕ for a region where $\Delta < 0$. Ex ($\overline{II}$)

$$ds^2 = -\frac{\Delta}{r^2}\,dv^2 \quad \text{(IEFC)}$$

$$= +\frac{|\Delta|}{r^2}\,dv^2 \longrightarrow \text{space like}$$

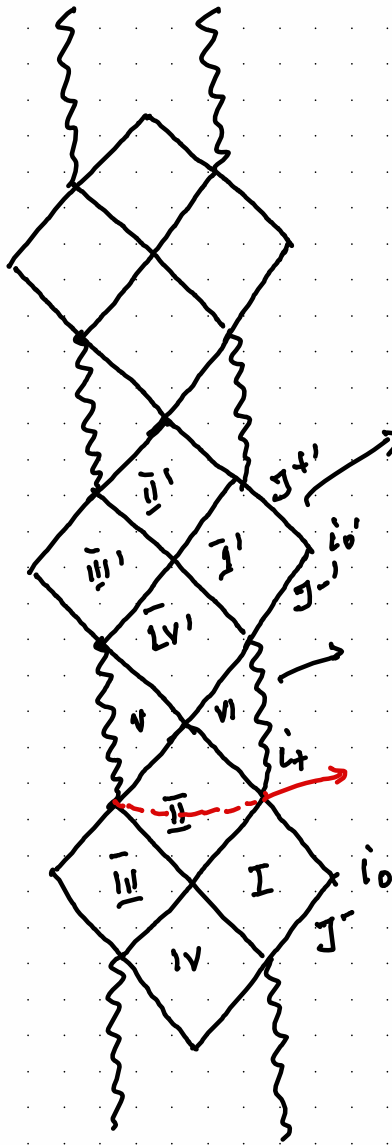
The proper length from $v=0$ to $v=-\infty$

$$\Rightarrow S = \int_{-\infty}^{0} \frac{|\Delta|^{1/2}}{r}\, dv = \frac{|\Delta|^{1/2}}{r} \int_{-\infty}^{0} dv = \infty .$$

Behind $r=r_+$ in $\bar{\text{II}}$, $\exists$ a spatial ∞.

(Since $V^{\pm}=0$ can be reached in finite proper time, these hypersurfaces are apart of spacetime)

At first glance it seems that CCC is violated here.



new asymptotic spacetime.

curvature singularity

spatial ∞ in $\underline{\overline{II}}$

The RN solution is actually unstable:

Let A cross the EH and enter $\underline{\overline{II}}$ and hovers there. B stays in $\underline{\overline{I}}$. B sends a pulse of energy at regular intervals. A recieves these signals in finite time. These signals get blue shifted so the $T_\mu{}^\mu$ content in $\underline{\overline{II}}$ grows. A tiny pulse) perturbation in $\underline{\overline{I}}$ gets amplified in $\underline{\overline{II}}$. The effect of this instability is that the Cauchy HS at spatial ∞ collapses to a horizon.
i.e RS $\longrightarrow$ Schwarzschild.

(BHs quickly neutralize themselves by plucking out an electron/anti-electron from the vacuum).

$\downarrow$

RN non-extremal

Let's look at extremal RN black holes:

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

$M = |Q|$

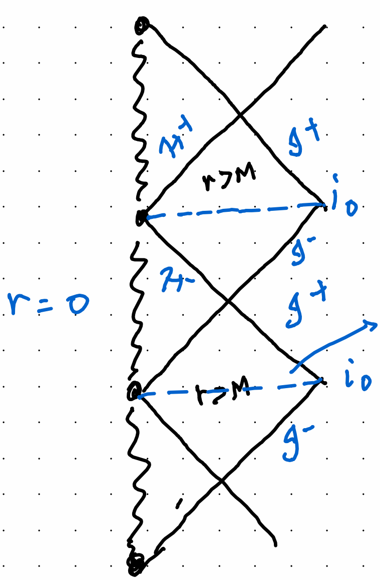
$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dt^2 + \left(1-\frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2$$

$$dr_* = \frac{dr}{\left(1-\frac{M}{r}\right)^2} \rightarrow r_* = r + 2M \ln\left|\frac{r-M}{M}\right| - \frac{M^2}{r-M}.$$

$u = t - r_*$, $v = t + r_*$

$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dv^2 + 2dv\,dr + r^2 d\Omega_2^2$$

which is smooth and extendible until $r=0$.



constant t hypersurfaces are no longer ER bridges.

but rather an ∞ throat b/w $r=0$ and spatial ∞.

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2$$

$$r = M(1+\lambda) \rightsquigarrow \quad \frac{M}{r} = \frac{1}{1+\lambda}$$

$$dr = M\, d\lambda$$

$$ds^2 = -\left(1 - \frac{1}{1+\lambda}\right)^2 dt^2 + \left(1 - \frac{1}{1+\lambda}\right)^{-2} dr^2 + r^2 d\Omega_2^2$$

$$\simeq \underbrace{-\lambda^2 dt^2 + \frac{M^2}{\lambda^2} d\lambda^2}_{AdS_2} + \underbrace{M^2 d\Omega_2^2}_{S_2}$$

NH of extremal RN BHs $\simeq AdS_2 \times S^2$

(Bertotti - Robinson)

A key result in AdS/CFT applications!

Can also introduce new radial coordinates

$$\rho = r - M$$

$$ds^2_{\text{Ext-RN}} = -H^{-2} dt^2 + H^2 \left(d\rho^2 + \rho^2 d\Omega_2^2 \right)$$

$$H = 1 + \frac{M}{\rho}.$$

special case

$$ds^2 = -H(x)^{-2} dt^2 + H(x)^2 \left(dx^2 + dy^2 + dz^2 \right)$$

$$\nabla^2 H = 0 \quad : \quad H = 1 + \sum_{i=N}^{N} \frac{M_i}{|x - x_i|}$$

3d Laplacian

N-RN BH's of mass M_i, $|Q_i| = M_i$

at x_i.

"multi center Black holes"

<u>RN</u> : → stability of spacetime

→ weak gravity conjecture ...