Towards Hodge Theoretic Characterizations of $2d$ Rational SCFTs

Abhiram Kidambi[1,2,a], Masaki Okada[1,b] and Taizan Watari[1,c]

[a]abhiram.kidambi@ipmu.jp, [b]masaki.okada@ipmu.jp , [c]taizan.watari@ipmu.jp

[1]*Kavli Institute for the Physics and Mathematics of the Universe (WPI), University of Tokyo, Kashiwa-no-ha 5-1-5, Kashiwa 277-8583, Japan*

[2] *Riemann Center for Geometry and Physics, Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover*

Abstract

The study of rational conformal field theories in the moduli space of conformal field theories is of particular interest since these theories correspond to points in moduli space where the algebraic and arithmetic structure are usually richer, while also being points where non–trivial physics occurs (such as in the study of attractor black holes and BPS states at rational points). This has led to various attempts to characterize and classify such rational points. In this paper, a conjectured characterization by Gukov–Vafa (Commun. Math. Phys. **246** (2004) 181) of rational conformal field theories whose target space is a Ricci flat Kähler manifold is analyzed carefully for the case of toroidal compactifications. We refine the conjectured statement as well as making an effort to verify it, using T^4 compactification as a test case. Seven common properties in terms of Hodge theory (including complex multiplication) have been identified for T^4-target rational conformal field theories. By imposing a subset of the seven properties, however, there remain $\mathcal{N} = (1,1)$ SCFTs that are not rational. Open questions, implications and future lines of work are discussed.

Contents

1 Introduction

Note to the reader: Although progress reported in this article is in string theory, and not mathematics, we still adopt the mathematical style of presentation using Theorem, Conjecture, Remark, Lemma etc. This makes it easier to refer to specific facts and/or arguments.

1.1 Relevant Background on String Theory and SCFT

$\mathcal{N} = (1, 1)$ supersymmetric non-linear sigma models in $1 + 1$ dimensions have a non-trivial moduli space when the target space M is a Ricci–flat Kähler manifold. These theories, which are superconformal field theories (SCFTs), define a compactification of Type II string theory. These SCFTs are *rational* CFTs (RCFT)[1] only at special points in the moduli space. It was observed (and conjectured) by Gukov–Vafa (GV) in [2] (see also [3, 4] and references therein) that the points in the moduli space where the SCFTs are rational may be characterized in terms of the period integrals on M and those on its mirror manifold using Hodge theory and number fields (to be more concrete, both M and its mirror are CM-type); see section 2.2 for more on the observation. Our current work presented here is inspired by these observations and we elaborate on the problem of characterizing these special points in the moduli space where the SCFTs are rational.

Giving (and establishing) such a characterization is a well-defined question in mathematical physics, which may be of interest in its own right. Apart from the GV conjecture, and some association with enhanced symmetry on worldsheet theory and/or effective field theory after compactification, our knowledge of rational SCFTs has been mostly construction based. Not much is known beyond the Gepner constructions, and lattice vertex operator algebras and their orbifolds. If the criteria for the SCFTs are proven to be something close to the one by the GV conjecture, that means that there are more rational SCFTs than those obtained by these constructional approaches. Those rational SCFTs will include the ones in the small-volume limit region in the moduli space; rational SCFTs will be an ideal (and a rare) tool to study how string theory captures geometry at the short-distance (high-energy)

[1]There are many ways to define a rational CFT. A simple way of explaining what an RCFT is to string theorists is that they have a finite number of primary fields or that their conformal blocks are finite dimensional (See [1] and references therein). Classifying and characterizing such rational CFTs is an important open problem in string theory. See section 2.1 for a dictionary between string theory and vertex operator algebra.

region in a situation where classical Einstein gravity is not a good approximation.[2]

Further progress can be expected in a couple of other directions, when the GV [2] observation is understood better, with more systematically constructed examples of rational SCFTs with geometric interpretations. For example, on the side of arithmetic geometry, it is known that complex analytic CM-type manifolds M are known to admit arithmetic models [5, 6, 7, 8], at least when M is either an abelian variety or a K3 surface, and some of the L-functions defined for these arithmetic models are expected to have modular transformation properties. It will be an interesting subject of research to explore relations between such modular objects in arithmetic geometry and $g = 1$ chiral correlation functions of the corresponding RCFTs in string theory. References [9, 10] studied that for the case where the target space is a CM elliptic curve. To carry out a similar study for abelian varieties and K3 surfaces, a better understanding of the relation between CM manifolds and RCFTs is necessary to get started.

Also in particle phenomenology, the work of GV [2] is of significance. In Type IIB Calabi–Yau orientifold compactifications, the gravitino mass and the cosmological constant are not generically much smaller than the Planck scale of the effective theory on $3+1-$dimensions due to non-zero fluxes [11]. When the complex structure of the Calabi–Yau threefold has period integrals characterized by number fields (such as in, [2, 12, 13]), then the gravitino mass can be much smaller than the Planck scale for a much larger fraction of flux configurations [14, 12, 15]. If the GV observation is true, then we may attribute particle phenomenologies such as electroweak gaugino dark matter, gauge coupling unification, and small gravitino mass, to large chiral algebra on the worldsheet theory, and not to a larger symmetry of the spacetime field theory.

1.2 Outline and Summary of the Paper

We begin in section 2.2 with a review of the conjectured connection between rational SCFTs and CM-type Hodge structures (Conj. 2.1) by following Ref. [2]. We do so while highlighting a few aspects in which the conjecture needs to be refined for its applications to various Ricci-flat Kähler manifolds. RCFTs among toroidal compactifications have been completely classified [16, 17, 18], so we use the information as test data in refining the GV conjecture. Moreover, we can also try to verify/find counter examples within toroidal compactifications. Results from the work of Meng Chen (MC) [19, Theorem 2.5 & Proposition 3.10] are vital to

[2]An open problem that is pertinent is here, and for which progress is desirable, is to also understand how close such rational points are to each other in the moduli space of SCFTs.

the underlying logic of the analysis performed in this study. The refined version of the GV conjecture is presented in Theorems 5.8 and 5.9 with auxiliary comments in section 6.

Sections 3 and 4.2 clarify subtleties raised in section 2.2 by studying T^4-target rational SCFTs. We will learn that RCFTs with T^{2n} target are associated not with complex tori with sufficiently many complex multiplications, but with CM-type abelian varieties (in section 3.1). We also see that we can always find a polarizable complex structure while demanding that the Hodge $(2,0)$ component of the $B-$field is absent (only for T^4, Prop. 3.4), and that there is always a mirror SCFT that allows a geometric interpretation (for T^4, Thm. 4.6). An observation that the Kähler form for a T^4-target rational CFT is in the algebraic part (non-transcendental part) of $H^2(T^4;\mathbb{Q})$ (Thm. 4.5) may also be interesting in its own right. We do not have evidence, however, that this is true for all the rational CFTs that are not T^4-target.

Sections 2.3–2.5 are, for the most part, review on related materials to be used in this article. Section 2.1 and Appendix A contain only textbook-level materials. We include them in this preprint so that readers with math background can get a rough sense from section 2.1 of what the authors mean by jargons such as compactification, CFT and $\mathcal{N} = (1,1)$ SCFT. Appendix A, on the other hand, collects some definitions, notations and useful facts in number theory and Hodge theory, for the convenience of some readers with background in string theory. The authors are happy to follow the advice from the editor/referees whether to keep Section 2.1 and Appendix A or to drop them from the manuscript.

2 Preliminaries

2.1 A Pertinent String Theory – VOA Dictionary

This subsection is absolutely not for a string theorist, but may be of some use to those who view themselves as members of the vertex operator algebra community. String theory may be viewed as machinery producing a conformal field theory (CFT) from a set of data associated with a geometry. This section 2.1 summarizes which data determine what in the machinery, and also explains the string theoretic terminology in this paper in the language of the vertex operator algebra community.

A bosonic conformal field theory (CFT) (as a countable noun):[3] it consists of data that

[3] We use the word "*bosonic conformal field theory*" in the same sense as "*vertex operator algebra*" of Ref. [20]. MC [21] uses *generalized vertex algebra* or *OPE algebra* for the same thing.

include (but are not exhausted by)

1. a pair of vertex operator algebras, one for the *left-mover* (i.e., one with the holomorphic local coordinates on a Riemann surface) and one for the *right-mover* (i.e., one with the anti-holomorphic local coordinates), and

2. a set of representations under the two mutually commuting vertex operator algebras.

The direct sum of all those representation spaces is called the *total Hilbert space*, $\mathcal{H}_{\rm tot}$. The states in the left-mover (resp. right-mover) vertex operator algebra must correspond to all the states of the total Hilbert space whose right-mover (resp. left-mover) conformal weights are zero. Such vertex operator algebras are called the left-mover (holomorphic) (resp. the right-mover (anti-holomorphic)) *chiral algebras of the CFT*.

When a string theorist refers to a bosonic CFT, it is often assumed implicitly that the both of the chiral algebras contain an operator called the energy-momentum tensor of conformal weight 2, that the central charge of the left-mover and the right-mover are not more than 26, that $\mathcal{H}_{\rm tot}$ has a positive definite Hermitian inner product, and that the partition function computed from $\mathcal{H}_{\rm tot}$ is modular invariant. The set of data of a bosonic CFT also includes an $\mathrm{End}(\mathcal{H}_{\rm tot})$-valued formal power series for any state in $\mathcal{H}_{\rm tot}$, not just for the states in the left-mover and right-mover chiral algebras. The authors do not intend to explain all the concepts in this paragraph, because not knowning them does not pose a problem in following the materials in this article.

A torus compactification associated with the data $(T^m; G, B)$: it is a bosonic CFT corresponding to the data $(T^m; G, B)$. Here,

1. $T^m = \mathbb{R}^m/\mathbb{Z}^{\oplus m} = \mathbb{R}^m/H_1(T^m;\mathbb{Z})$ is a real m-dimensional torus (a manifold) with a set of real coordinates $(X^I) = (X^1, X^2, \cdots, X^m) \in \mathbb{R}^m$.

2. G is a Riemannian metric on T^m that remains constant under translations in $\mathbb{R}^m$, and

3. B a closed 2-form on T^m, referred to as the B-field.

In the language of string theory, G is the vacuum metric and B the vacuum B-field configuration, to be precise, but we will omit the word 'vacuum' for brevity in this article.

For any bosonic CFT (i.e., torus compactification) for $(T^m; G, B)$, both the left-mover and right-mover chiral algebras contain the direct sum of m copies of the Heisenberg Lie algebra. Different choices of the data G, B as above correspond to different choices of the set of representations of the $m+m$ Heisenberg Lie algebras. The set of choices of such a set of representatons of the $m+m$ Heisenberg Lie algebras (and hence the set of data $(T^m; G, B)$) forms a moduli space. For further details, we refer the reader to [20].

A bosonic CFT is rational when both the left-mover (holomorphic) chiral algebra and the right-mover (anti-holomorphic) chiral algebra are rational, meaning that both of the two vertex operator algebras have a finite number of distinct irreducible modules [22, p.90].

A torus (T^m) compactification for G and B **is rational** if both of the holomorphic and anti-holomorphic chiral algebras are larger than m copies of the Heisenberg Lie algebras to the extent that each of the chiral algebras becomes the vertex operator algebra of a rank-m even positive definite lattice.

An $\mathcal{N} = (1,1)$ SCFT (as a countable noun) **associated with the data** $(M; G, B)$, where M is a real manifold, G a Ricci-flat Riemannian metric on M, and B a closed 2-form on M: It is an $\mathcal{N} = (1,1)$ SCFT (a countable noun) determined uniquely in string theory by the data $(M; G, B)$. An $\mathcal{N} = (1,1)$ SCFT is a set of information that includes (but is not exhausted by)

1. two vertex operator superalgebras (one for left-mover (holomorphic) and the other for right-mover (anti-holomorphic)), each one of which contains an $\mathcal{N} = 1$ superconformal algebra, and

2. a set of representations under the mutually (anti-)commuting algebras.

The states in the left-mover (resp. right-mover) vertex operator superalgebra must correspond to all the states in $\mathcal{H}_{\rm tot}$ whose right-mover (resp. left-mover) conformal weight is zero. These vertex operator superalgebras are called the left-mover and right-mover *chiral superalgebra of the SCFT*. This definition does not rule out an $\mathcal{N} = (1,1)$ SCFT whose chiral superalgebra contains an $\mathcal{N} = 2$ (or more) superconformal algebra.

If there exists a complex structure I for a Riemannian manifold $(M; G)$ such that G is compatible with I and $(M; G, ; I)$ is Kähler, then the $\mathcal{N} = (1,1)$ SCFT for the data $(M; G, B)$, with B as above, has a special property. Each one of the left-mover and right-mover chiral superalgebras contains an $\mathcal{N} = 2$ superconformal algebra. In fact, there is a unique way to specify the vertex operator with conformal-weight 1 from the data I such that the $\mathcal{N}=1$ superconformal algebra is enhanced to an $\mathcal{N}=2$ superconformal algebra. We refer the reader to [23] for more information.

An $\mathcal{N}=(1,1)$ SCFT is rational when both the left-mover and right-mover chiral superalgebras have finitely many distinct irreducible modules. It is clear that the rationality of an $\mathcal{N}=(1,1)$ SCFT depends on the entire chiral superalgebra determined by the data $(M;G,B)$, and not by the $\mathcal{N}=(2,2)$ superconformal algebra determined by the data $(M;G,B;I)$. It is also a known fact that the $\mathcal{N}=(1,1)$ SCFT for $(T^m;G,B)$ is rational if and only if a torus compactification for (i.e., bosonic CFT for) the same set of data $(T^m;G,B)$ is rational.

2.2 The Gukov–Vafa Conjecture

Amongst bosonic CFTs with torus (T^m) as target, RCFTs have been identified completely [17, 18, 16]. In the case of $m=1$, for example, the moduli space of bosonic CFTs is $\mathbb{R}_{>0}$ and is physically parametrized by the radius-squared of the target space S^1. Amongst such bosonic CFTs, only those CFTs for which the radius-squared (in units of the string length) is a rational number are rational. So, the subset $\mathbb{Q}_{>0} \subset \mathbb{R}_{>0}$ classifies all RCFTs for the case of $m=1$.

However, not much is known for cases other than torus compactifications. Certain explicit constructions such as the Gepner models are known, but it is not known how many more rational CFTs or SCFTs exist.

The GV conjecture [2] was formulated in an attempt to ascertain where one encounters a rational SCFT in the moduli space of $2d$ $\mathcal{N}=(1,1)$ SCFTs obtained as a non-linear-sigma model of a manifold M that admits a Ricci-flat Kähler metric.

Conjecture 2.1. [2, §7] Consider a $2d$ $\mathcal{N}=(1,1)$ SCFT obtained as a non-linear sigma model of $(M;G,B)$; here, M is a real $2n$-dimensional manifold, B a closed $2-$form on M, and G a Riemannian metric on M that can be Ricci-flat and Kähler under some suitable complex structure I. The SCFT is rational if and only if the following conditions are satisfied:

1. the rational Hodge structure on the cohomology groups of M is of CM-type,
2. the rational Hodge structure on the cohomology groups of the mirror manifold W of M is also CM-type,
3. the CM fields of both are isomorphic.

Conjecture 2.1 presented as above is a slight modification of the conjecture presented in Ref. [2], although its scientific merit is not compromised. •

We first repeat the justification argument for Conjecture 2.1 for the convenience of a less initiated reader. We then later discuss the refinement of the statement of Conjecture 2.1. For a physicist-friendly overview of theory of complex multiplication covering more than elliptic curves, readers are referred to the appendix of [15]; in the case of elliptic curves to get started, see [3, 4].

The first piece of evidence in support of Conjecture 2.1 above is the case of T^2 compactification. Consider a T^2 compactification associated with the data $(T^2; G, B)$, where G and B are a constant Riemannian metric and a constant 2-form on T^2, respectively. There is a unique complex structure I with which the metric G is compatible. Let (the $\mathrm{SL}(2;\mathbb{Z})$-orbit of) τ be the complex structure parameter of a complex $g=1$ Riemann surface $M = (T^2; I) = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, and $\rho := \int_{T^2} (B + i\omega)/[(2\pi)^2\alpha']$ be the complexified Kähler parameter; $\omega = \omega(-,-) := 2^{-1}G(I-,-)$ is the Kähler form. It is then known that the CFT is rational if and only if both $\mathbb{Q}(\tau)$ and $\mathbb{Q}(\rho)$ are degree-2 extension fields over $\mathbb{Q}$, and are isomorphic to each other [3, 4]. The condition that $[\mathbb{Q}(\tau):\mathbb{Q}] = 2$ (resp. $[\mathbb{Q}(\rho):\mathbb{Q}] = 2$) is equivalent to the condition that the rational Hodge structure on $H^1(M;\mathbb{Q})$ (resp. $H^1(W;\mathbb{Q})$, where W is the mirror manifold isomorphic to $\mathbb{C}/(\mathbb{Z}+\rho\mathbb{Z})$) is of CM-type. Those observations in this example have been abstracted to become the statement of Conjecture 2.1 above.

Let us consider another example: the $\mathbb{Z}_5$-orbifold of the tensor product of five copies of the $2d$ $\mathcal{N} = (2,2)$ minimal models with the central charges $c_L = c_R = 3k/(k+2)$ and $k = 3$. This SCFT is rational, and is interpreted as a $2d$ non-linear sigma model whose target space is a quintic Calabi–Yau threefold with a very special complex structure and a very special complexified Kähler parameter. The complexified Kähler parameter is chosen at the small volume limit within the complex 1−dimensional moduli space, and the complex structure parameter is chosen at the Fermat point[4] of the complex 101−dimensional moduli space. The cohomology group $H^3(W;\mathbb{Q})$ of the mirror manifold W is 4-dimensional over $\mathbb{Q}$, and is of CM–type, where the CM–field is $\mathbb{Q}(\zeta_5)$, a cyclotomic extension field over $\mathbb{Q}$ generated by a primitive 5th root of unity ζ_5. The cohomology group $H^3(M;\mathbb{Q})$ also contains a rational Hodge substructure that is 4−dimensional over $\mathbb{Q}$, level−3, and is of CM–type; the CM–field is $\mathbb{Q}(\zeta_5)$ on this substructure. Various jargons pertaining to Hodge structure are explained

[4]The 101−dimensional moduli space of complex structure corresponds to choosing an arbitrary homogeneous function $F(\Phi_{i=1,\cdots,5})$ of degree-5 on $\mathbb{P}^4$ that defines a threefold M through $M = \{[\Phi_i] \in \mathbb{P}^4 \mid F(\Phi) = 0\} \subset \mathbb{P}^4$. The Fermat point in the moduli space corresponds to the choice $F = \sum_{i=1}^{5} (\Phi_i)^5$.

in the appendix of [15], or any textbook or lecture note on Hodge theory by mathematicians. This example indicates that GV's Conjecture 2.1 has been generalized from the case of T^2 compactification in a proper way.

We are yet to verify or refute this conjecture, and that is what we do in section 5. Such an effort also lets us notice that the statement of the conjecture needs to be refined to be verified, as we see below in Discussions 2.2, 2.3, 2.4 and 2.5.

2.2. Let M be a Ricci-flat Kähler manifold of complex dimension n; the cohomology group $H^n(M;\mathbb{Q})$ is endowed with a rational Hodge structure by the complex structure of M. The rational Hodge structure on $H^n(M;\mathbb{Q})$ is not necessarily simple, but has a decomposition into simple Hodge substructures

$$H^n(M;\mathbb{Q}) \cong \oplus_{a\in A}[H^n(M;\mathbb{Q})]_a \ . \tag{2.1}$$

Let $D_a := \mathrm{End}([H^n(M;\mathbb{Q})]_a)^{\mathrm{Hdg}}$ be the algebra of Hodge endomorphisms of a simple component $[H^n(M;\mathbb{Q})]_a$; it is always a division algebra.[5] The endomorphism algebra of $H^n(M;\mathbb{Q})$ is of the form

$$\mathrm{End}(H^n(M;\mathbb{Q}))^{\mathrm{Hdg}} \cong \oplus_\alpha M_{n_\alpha}(D_\alpha), \tag{2.2}$$

where the simple components $a \in A$ are grouped into those with isomorphic D_a's and a common level, and the equivalence classes are labeled by α's; n_α is the number of simple components (a's) in an equivalence class α.

For example, in the case of the Fermat quintic Calabi–Yau threefold M, the 204-dimensional vector space $H^3(M;\mathbb{Q})$ has a decomposition into simple rational Hodge substructures [24, §3],

$$[H^3(M;\mathbb{Q})]_{\ell=3} \oplus \left(\oplus_{a=1}^{50}[H^3(M;\mathbb{Q})]_{\ell=1,a}\right), \tag{2.3}$$

and each one of the components is of 4−dimensional over $\mathbb{Q}$, supporting a rational Hodge substructure of level−1 (with the exception of the first component whose Hodge substructure is of level 3) with the endomorphism field $D_\alpha \cong \mathbb{Q}(\zeta_5)$.

Back to the general case, the Hodge structure on $H^n(M;\mathbb{Q})$ is said to be *of CM-type* if and only if the Hodge structure on individual substructures on $[H^n(M;\mathbb{Q})]_a$ are of CM-type. All the division algebras D_a are then fields, with $[D_a : \mathbb{Q}] = \dim_{\mathbb{Q}}[H^n(M;\mathbb{Q})]_a$. So, the Hodge

[5] See Definition A.2 and Notation A.7.

structure on $H^n(M;\mathbb{Q})$ is of CM-type in the case that M is the Fermat quintic threefold. There are, however, Kähler manifolds where the Hodge structure is of CM-type on the level-n component (the simple component containing the $(n, 0)$ Hodge component; the notion of *level* is explained, e.g., in [15, App. B]), but is not of CM-type in other simple components. We should therefore remain open minded as to whether the first two conditions in Conjecture 2.1 should be imposed on some of the simple components of $H^n(M;\mathbb{Q})$ and $H^n(W;\mathbb{Q})$, or on all of their simple components. Examples studied in [2] are not sufficient to resolve this difference in the conditions.[6]

The third condition in Conjecture 2.1 refers to the fields of CM-type Hodge structures of M and the mirror manifold W. An easy way to make sense of this condition is to think of them as the endomorphism fields of the unique simple component of $H^n(M;\mathbb{Q})$ and $H^n(W;\mathbb{Q})$ containing the Hodge $(n, 0)$ component, the level-n simple component. If one should require other simple components to be of CM-type, as discussed before, then one may also have to refine the the third condition; whether it is read as an isomorphism of the CM fields of the level-n components on both sides, or as isomoprhisms of some pairs of simple components of $H^*(M;\mathbb{Q})$ and $H^*(W;\mathbb{Q})$. In general, $\dim_{\mathbb{Q}}[H^n(M;\mathbb{Q})]$ is not necessarily equal to $\dim_{\mathbb{Q}}[H^n(W;\mathbb{Q})]$, so there is no natural choice of pairs of simple components besides the pair of the level-n components.

2.3. For a Ricci-flat Kähler manifold M, its Hodge diamond can be non–zero not only in the vertical diagonal terms ($h^{k,k}$ with $k = 0, \cdots, n$) and horizontal diagonal terms ($h^{p,n-p}$ with $p = 0, \cdots, n$), but also in the off–diagonal terms. Certainly all the off–diagonal terms are zero when M is an elliptic curve, K3 surface, or a Calabi–Yau threefold. But $h^{q,0}$ with $q \neq 0, n$ can be non-zero when M is

- a complex torus of $n \geq 2$ dimensions, or
- a hyper-Kähler manifold of real 8-dimensions and higher, or
- a product of Ricci-flat Kähler manifolds one of which is either a complex torus or a hyper-Kähler manifold.

Moreover, there are Calabi–Yau fourfolds where all the $h^{q,0}$'s are zero for $q = 1, \cdots, 3$, but the off-diagonal term $h^{2,1}$ is non-zero (see [27, (90)] for a class of toric hypersurface fourfolds where $h^{2,1} \neq 0$).

[6] Calabi–Yau threefolds of the form of Borcea–Voisin orbifolds [25, 26] will be a good testing ground in resolving this issue. To work on this class of cases, however, we should work on K3 surfaces first.

The authors of this paper are not aware of a proof indicating that the Hodge structure on $H^k(M;\mathbb{Q})$ with $k \neq n$ is always of CM-type when one just requires that the Hodge structure on $H^n(M;\mathbb{Q})$ is of CM-type (see Rmk. 5.3). We then face a question whether we should read the conditions in Conjecture 2.1 as that for $H^n(M;\mathbb{Q})$ and $H^n(W;\mathbb{Q})$ (the vertical part of the Hodge diamond of M), or that for all the cohomology groups including the off-diagonal parts of the cohomology group of M.

2.4. Consider a case where the target space $(M;G)$ is either a torus T^{2n} of real $2n$ dimensions with $n \geq 2$, or a hyper-Kähler manifold. On one hand, for such a smooth manifold M and a Riemannian metric G on it, there is a continuous freedom in choosing a complex structure I with which the metric G is compatible.

On the other hand, a $2d$ non-linear sigma model is specified by only the data $(M;G)$, without referring to a complex structure on M. Whether the SCFT is rational or not should therefore be a property of $(M;G)$, not of the data $(M;G;I)$.

Since Conjecture 2.1 tries to characterize rational SCFTs by using a Hodge structure, there is no way of interpreting the conditions there without choosing a complex structure. If the conjecture is to be applicable for the class of manifolds we are referring to here, then we should read the conditions and characterizations in Conjecture 2.1 either as those for arbitrary I with which the metric is compatible (this is not a good guess as we will see in section 3.1), or as those for a class of I's that should be specified more carefully.

2.5. The statement of Conjecture 2.1 is written by referring to a mirror manifold W. It is not always true, however, that an $\mathcal{N} = (2,2)$ SCFT as a non-linear sigma model with a Ricci-flat Kähler M as the target space has a mirror-equivalent $\mathcal{N} = (2,2)$ SCFT that can be interpreted as a non-linear sigma model of another Ricci-flat Kähler manifold W. Even when there is, it is not guaranteed that there is a unique choice of W.

It is an interesting question whether there is always such a mirror manifold when the M-target $\mathcal{N} = (2,2)$ SCFT is rational. Would it be an improvement if Conjecture 2.1 is stated without referring to a mirror manifold?

In this article, we work on the cases with $M = T^{2n}$, most intensively with $M = T^4$. The experimental data collected in this article do not help in resolving the issue raised in Discussion 2.2, but will shed some light on the issues raised in 2.4 and 2.5. The authors admit that they have not made all possible efforts imaginable in exploiting the experimental data to clarify the issues in 2.3 and 2.5.

2.3 Rational CFTs with Torus Target

Since RCFTs in torus compactifications have been completely classified, we may use the established results to refine and test Conjecture 2.1. In this section 2.3, we quote results from Ref. [16, 17] relevant to our analysis.

Proposition 2.6. ([16] and [17, Lemma 4.5.1]) *Let $T^m = \mathbb{R}^m/\mathbb{Z}^{\oplus m}$ be a real m-dimensional torus with a smooth structure, X^Is with $I = 1, \cdots, m$ a set of coordinates of $\mathbb{R}^m$ with periodicity ΔX (i.e., $X^I \sim X^I + \Delta X$), let $G = G_{IJ} dX^I \otimes dX^J$ be a constant Riemannian metric on T^m (i.e., $G_{IJ} \in \mathbb{R}$ are independent of the coordinates X^K's), and $B = 2^{-1} B_{IJ} dX^I \wedge dX^J$ a 2-form on T^m where B_{IJ} are independent of the coordinates.*

The bosonic CFT for the data $(T^m; G, B)$ is rational if and only if [7]

$$\left(\frac{\Delta X}{(2\pi)\sqrt{\alpha'}}\right)^2 G_{IJ} \in \mathbb{Q}, \qquad \left(\frac{\Delta X}{(2\pi)\sqrt{\alpha'}}\right)^2 B_{IJ} \in \mathbb{Q}. \tag{2.4}$$

The condition for the $\mathcal{N} = (1,1)$ SCFT associated with the data $(T^m; G, B)$ to be rational is also the same as above.

We are interested in the cases with $m = 2n$, when there is a possibility of introducing a complex structure on the target space T^m. The author of [17] has further derived this

Corollary 2.7. [17, Thm. 4.5.5] *Let $(T^{2n}; G, B)$ be a set of data for which the (S)CFT is rational. Then there exists a surjective homomorphism $\varphi : T^{2n} \cong \mathbb{R}^{2n}/\mathbb{Z}^{\oplus 2n} \longrightarrow \prod_{a=1}^{n} \mathbb{C}/(\mathbb{Z} + \tau_a \mathbb{Z})$ with respect to the abelian group law on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$, where each one of $\mathbb{C}/(\mathbb{Z} + \tau_a\mathbb{Z})$ is a CM elliptic curve (i.e., $[\mathbb{Q}(\tau_a) : \mathbb{Q}] = 2$), and there is a metric on $\mathbb{C}/(\mathbb{Z} + \tau_a\mathbb{Z})$, given by $ds^2 = g_a(du^a \otimes d\bar{u}^{\bar{a}} + \text{h.c.})$ with*[8] *$g_a \in \mathbb{Q}$ so that the pull-back of the metric ds^2 by φ agrees with the metric G on T^{2n}.*

This result lends support towards the justification of the GV Conjecture 2.1 in the following sense. Firstly, there is already an implicit choice of complex structure I_0 on $\prod_a \mathbb{C}/(\mathbb{Z} + \tau_a\mathbb{Z})$, with which the metric ds^2 is compatible. The metric G is compatible with the complex structure $I = \varphi^*(I_0)$. The complex torus $(T^{2n}; I)$ is of CM–type since $\prod_a \mathbb{C}/(\mathbb{Z}+\tau_a\mathbb{Z})$ is of CM–type. The metric $g_a \in \mathbb{Q}$ should be split into $g_a = \mathrm{Im}(\rho_a)/\mathrm{Im}(\tau_a)$ so that $\mathrm{Im}(\rho_a)$ parametrizes the volume of $\mathbb{C}/(\mathbb{Z}+\tau_a\mathbb{Z})$. It also follows that $\mathbb{Q}(i\mathrm{Im}(\rho_a)) \cong \mathbb{Q}(\tau_a)$.

[7] The author of [17] adopts the convention $\Delta X = 2\pi R$, $R = \sqrt{\alpha'}$ and $\alpha' = 2$. We will use the convention $\Delta X = 2\pi\sqrt{\alpha'}$ throughout this article. The metric and B-field satisfying (2.4) are therefore said to be *rational*.

[8] u^a with $a = 1, \cdots, n$ are the complex coordinates of the a-th elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau_a\mathbb{Z})$, which has the periodicity $u^a \sim u^a + 1 \sim u^a + \tau_a$.

This observation alone still falls short of resolving the issue raised in 2.5. It also remains to be an open question whether the class of complex structures of the form $I = \varphi^*(I_0)$ are all those where a GV–like statement holds true (this issue was raised in 2.4). We will discuss those issues in sections 3 and 4, before examining whether Conjecture 2.1 holds true or not in section 5.

2.4 Horizontal and Vertical Generalized Complex Structures

Generalized complex structure/Kähler structure and their relation to mirror symmetry are reviewed in this section 2.4. For a reader familiar with the work in such references as [28, 20, 29, 23, 30], this section 2.4 does not do anything more than preparing notations.

2.8. Horizontal generalized Hodge structure on $H^*(T^{2n};\mathbb{Q})$: Let I and B' be a complex structure and a $\mathbb{R}$-valued closed 2-form on T^{2n}, respectively. Using I and B', a linear operator $\mathcal{I}_{B'}$ on the space of sections of $T(T^{2n}) \oplus T^*(T^{2n})$ is introduced:

$$\mathcal{I}_{B'} : (\partial_{X^I}, dX^I) \longmapsto (\partial_{X^L}, dX^L) \begin{pmatrix} \delta^L_K & \\ B'_{LK} & \delta^K_L \end{pmatrix} \begin{pmatrix} I^K_J & \\ & (I^{-1})^J_K \end{pmatrix} \begin{pmatrix} \delta^J_I & \\ -B'_{JI} & \delta^I_J \end{pmatrix}, \quad (2.5)$$

where we have chosen a basis $\{\partial_{X^I}\}$ and $\{dX^I\}$ for the tangent and cotangent spaces at each point on T^{2n}; $B' =: 2^{-1}B'_{IJ}dX^I \wedge dX^J$ is the usual convention (see [31, App. B.4]). In the absence of B', $\mathcal{I}_{B'}$ multiplies $(+i)$ to holomorphic tangent vectors and (0,1)-forms, and multiplies $(-i)$ to anti-holomorphic tangent vectors and (1,0)-forms. The operator $\mathcal{I}_{B'}$ in (2.5) is an example[9] of a *generalized complex structure* on T^{2n} [28, 20].

Let $\Lambda := H_1(T^{2n};\mathbb{Z}) \oplus H^1(T^{2n};\mathbb{Z})$, and q be the bilinear form given by $q(\partial_{X^I}, \partial_{X^J}) = q(dX^I, dX^J) = 0$ and $q(\partial_{X^I}, dX^J) = \delta_I^J$. The integral Hodge structure introduced by $\mathrm{diag}(I, (I^{-1})^T)$ on $\Lambda_\mathbb{R} := \Lambda \otimes \mathbb{R}$ has been deformed by the 2-form B' to be $\mathcal{I}_{B'}$. The deformed version can be expressed by a representation of $\mathrm{U}(1) \cong S^1$ given as follows: First, noting that

$$(\mathcal{I}_{B'})^T \cdot q \cdot \mathcal{I}_{B'} = q, \quad (2.6)$$

we choose an element $iX_{I,B'}$ of the Lie algebra $\mathfrak{so}(\Lambda_\mathbb{R}, q)$ acting on $\Lambda_\mathbb{R}$:

$$\mathcal{I}_{B'} = \exp\left[\frac{\pi}{2} iX_{I,B'}\right]. \quad (2.7)$$

[9]See Refs. [29, 30] for how the notion of a generalized complex structure is defined for a general real manifold M not necessarily a torus.

With this (c.f. A.6),

$$h_{I,B'} : S^1 \ni e^{i\alpha} \longmapsto \exp\left[i\alpha X_{I,B'}\right] \in \mathrm{GL}(\Lambda_{\mathbb{R}}). \tag{2.8}$$

The vector space $\Lambda \otimes \mathbb{C}$ splits into the $h_{I,B'}(e^{i\alpha}) = e^{i\alpha}$ representation space (containing (0,1)-forms), and $h_{I,B'}(e^{i\alpha}) = e^{-i\alpha}$ representation space (containing (1,0)-forms), but this deformed version of the "Hodge" decomposition is not something we wish to think as a pure Hodge structure of some given weight any more.

The operator $\mathcal{I}_{B'}$ and $\exp[i\alpha X_{I,B'}]$ are elements of the Lie group $\mathrm{SO}(\Lambda_{\mathbb{R}}, q)$; the S^1 subgroup is denoted by $S^1_{I,B'}$. Now, we may think of the spinor representation ρ_{spin} of the $\mathrm{SO}(\Lambda_{\mathbb{R}}, q)$ group and its restriction to the $S^1_{I,B'}$ subgroup; the representation $\rho_{\text{spin}}|_{S^1_{I,B'}}$ of $S^1_{I,B'}$ is denoted by $\rho_{\text{spin}}(h_{I,B'})$. The ρ_{spin} representation space of $\mathrm{SO}(\Lambda_{\mathbb{R}}, q)$ has an isomorphism[10] with $H^*(T^{2n}; \mathbb{Q}) \otimes \mathbb{R}$, so we always use this interpretation freely. The spinor representation of the SO group splits into the two irreducible representations, one on $H^{\text{even}}(T^{2n}; \mathbb{Q}) \otimes \mathbb{R}$ and the other on $H^{\text{odd}}(T^{2n}; \mathbb{Q}) \otimes \mathbb{R}$.

The representation $\rho_{\text{spin}}(h_{I,B'})$ of $S^1_{I,B'}$ introduces something similar (but not quite) to the rational mixed Hodge structure[11] on $H^{\text{even}}(T^{2n}; \mathbb{Q})$ and $H^{\text{odd}}(T^{2n}; \mathbb{Q})$. The k-th cohomology

[10] Here is a brief note on the convention. Let $\Lambda_{\mathbb{R}}$ be a $2m$-dimensional vector space, and q its symmetric bilinear form of signature (m, m). Suppose that $L_{\mathbb{R}} \subset \Lambda_{\mathbb{R}}$ is an isotropic subspace of m-dimensions, and $\{e_{I=1,\cdots,m}\}$ its basis. Then set $L'_{\mathbb{R}} := [L^{\perp}_{\mathbb{R}} \subset \Lambda_R]$, and choose a basis $\{e_{m+I;I=1,\cdots,m}\}$ of $L'_{\mathbb{R}}$. The representation space of ρ_{spin} of $\mathrm{SO}(\Lambda_{\mathbb{R}}, q)$ is constructed as follows. The Clifford algebra is given by

$$\mathrm{Cliff}(\Lambda_{\mathbb{R}}, q) := \mathbb{R}[x_{1,\cdots,m}, x_{m+1,\cdots,m+m}]/(\{x_I, x_J\}, \{x_{m+I}, x_{m+J}\}, \{x_I, x_{m+J}\} - 2q(e_I, e_{m+J})),$$

and the representation space we want is the left-ideal $\mathfrak{a}_L$ of $\mathrm{Cliff}(\Lambda_{\mathbb{R}}, q)$ generated by the element $x_{1\cdots m} := x_1 x_2 \cdots x_m$. See [28, §3.2.1] for more information.

When $\Lambda_{\mathbb{R}} = H_1(T^m; \mathbb{R}) \oplus H^1(T^m; \mathbb{R})$, one may set the maximal isotropic subspace $L_{\mathbb{R}}$ to be $H_1(T^m; \mathbb{R})$. The isomorphism $\mathfrak{a}_{H_1(T^m;\mathbb{R})} \cong H^*(T^m; \mathbb{R})$ is given by assigning $x_{m+I_1} x_{m+I_2} \cdots x_{m+I_k}(x_{1\cdots m})$ to $dX^{I_1} \wedge dX^{I_2} \wedge \cdots \wedge dX^{I_k} \in H^k(T^m; \mathbb{R})$.

[11] A mixed rational Hodge structure generalizes the pure rational Hodge structure on $H^k(M; \mathbb{Q})$ with a fixed k, when M is not necessarily a compact smooth Kähler manifold, but a possibly open and singular variety. A *mixed rational Hodge structure* [32] on a vector space $V_{\mathbb{Q}}$ over $\mathbb{Q}$ consists of one decreasing filtration $F^\bullet$ (called *Hodge filtration*), where $V \otimes \mathbb{C} \supset \cdots \supset F^p \supset F^{p+1}$, and one *increasing* filtration $W_\bullet$ (called *weight filtration*), where $W_m \subset W_{m+1} \subset \cdots \subset V_{\mathbb{Q}}$. The component $(F^p W_m \otimes \mathbb{C}/F^p W_{m-1} \otimes \mathbb{C}) \cap \overline{(F^q W_m \otimes \mathbb{C}/F^q W_{m-1} \otimes \mathbb{C})}$ is regarded the $(p, m-p)$ component.

Technically, it is not impossible to think of the generalized rational Hodge structure in Def. 2.9 as a mixed rational Hodge structure. To get started, note that the difference between a decreasing filtration $W^\bullet_h$ in Def. 2.9 and an increacing filtration $W_\bullet$ of a mixed rational Hodge structure is relatively minor. One may set a weight filtration of a mixed Hodge structure by $W_m := W^{2n-m}_h$; we would have to think of $2n$-forms as weight $(m = 0)$ then, but we could close our eyes to that.

It is not impossible to use the S^1 representation h of a generalized Hodge structure to introduce the decreasing filtration $F^\bullet$ of a mixed Hodge structure; an idea that comes to the minds of the authors is to group $H^k(T^{2n};\mathbb{Q})\otimes\mathbb{R}$ of $H^*(T^{2n};\mathbb{Q})\otimes\mathbb{R}$ alone is not regarded as a representation space of the $S^1_{I,B'}$ subgroup when $B'\cdot I-(I^{-1})^T\cdot B'\propto(B')^{(2,0)}-(B')^{(0,2)}\neq 0$. But there is still a filtration structure

$$\{0\}\subset W_h^{2n}\ \subset W_h^{2n-2}\subset\cdots\subset W_h^2\subset W_h^0=H^{\mathrm{even}}(T^{2n};\mathbb{Q}), \tag{2.9}$$
$$\{0\}\subset W_h^{2n-1}\ \subset W_h^{2n-3}\subset\cdots\subset W_h^3\subset W_h^1=H^{\mathrm{odd}}(T^{2n};\mathbb{Q}), \tag{2.10}$$

of vector subspaces over $\mathbb{Q}$, where $W_h^k\otimes_{\mathbb{Q}}\mathbb{R}$ supports a sub–representation of $\rho_{\mathrm{spin}}(h_{I,B'})$; here,

$$W_h^{2n-2\ell}:=\oplus_{m=0}^{\ell}H^{2n-2m}(T^{2n};\mathbb{Q}),\qquad W_h^{2n-1-2\ell}:=\oplus_{m=0}^{\ell}H^{2n-1-2m}(T^{2n};\mathbb{Q}). \tag{2.11}$$

The induced representation of $\rho_{\mathrm{spin}}(h_{I,B'})$ on $(W_h^k/W_h^{k+2})\otimes\mathbb{R}$ agrees with the representation of $S^1_{I,B'}$ that describes the pure Hodge structure of weight-k on $H^k(T^{2n};\mathbb{Q})$ obtained from the complex structure I alone. $S^1_{I,B'}\ni e^{i\alpha}\longmapsto e^{-i\Delta\alpha}$ with $\Delta=(p-q)$ on the Hodge (p,q) component (cf A.6).

It is therefore motivated to introduce the following notion.

Definition 2.9. On a vector space $V_{\mathbb{Q}}$ over $\mathbb{Q}$, one may introduce a set of data $(h,W^\bullet)$ called a *generalized rational Hodge structure* whose properties are specified below. $W^\bullet$ is a decreasing filtration, a sequence of vector subspaces over $\mathbb{Q}$, $\{0\}\subset\cdots\subset W^1\subset W^0=V_{\mathbb{Q}}$, and h is a representation of S^1, $h:S^1\ni e^{i\alpha}\longmapsto h(e^{i\alpha})\in\mathrm{GL}(V_{\mathbb{Q}}\otimes\mathbb{R})$, where each one of the subspaces $W^k\otimes\mathbb{R}$'s supports a sub–representaton of h. We call the $h(e^{i\alpha})=e^{-i\alpha\Delta}$ subspace of $W^k\otimes\mathbb{C}$ the *charge* Δ *component of* $W^k\otimes\mathbb{C}$. One might be interested in introducing the notion that a generalized rational Hodge structure is *polarizable*, but the authors do not feel fully ready to do so.[12]

set $F^\Delta:=\oplus[\text{charge}\geq\Delta]$. The range of (Δ,m) with a non-zero $h^{\Delta,m-\Delta}$ in this rational mixed Hodge structure (from $(\rho_{\mathrm{spin}}(h_{I,B'}),W_h^\bullet)$) on $H^*(T^{2n};\mathbb{Q})$ is quite different from the mixed rational Hodge structure on $H^n(M;\mathbb{Q})$ of a non-compact and/or singular complex n-dimensional variety M. The range becomes the conventional one when we set a dictionary $p=(m+\Delta)/2$. Conversely, the information in the Hodge filtration of a mixed Hodge structure on $H^k(M;\mathbb{Q})$ can also be translated into the language of the S^1 representation; a differential form $(\wedge^p dz)(\wedge^{k-p}d\bar z)/(\prod_{i=1}^{p+q-k}z_i)$ generating the component $[H^k(M;\mathbb{Q})]^{p,q}$ (not necessarily $p+q=k$) is assigned a charge $\#[dz]-\#[d\bar z]-\#[\text{poles}]=p-(k-p)-(p+q-k)=p-q$.

Despite the similarity between the mixed and generalized Hodge structures at the technical level, both structures are based on completely different geometric intuitions. It does not seem possible for a set of data $(\rho_{\mathrm{spin}}(h_{I,B'}),W_h^\bullet)$ to think of something like $F^p\sim[\text{charge}\geq\Delta=(2p-m)]\subset(W_m=W_h^{2n-m})\otimes\mathbb{C}$ consistently with varying choice of m.

[12]An idea will be to generalize the notion of a polarizable rational Hodge structure of a Kähler manifold

So, for a closed (constant) 2-form B' and a complex structure I on T^{2n}, the set of data $(\rho_{\text{spin}}(h_{I,B'}), W_h^\bullet)$ on $H^*(T^{2n};\mathbb{Q})$ introduces a generalized rational Hodge structure. We call it the *horizontal generalized rational Hodge structure* for $(B';I)$.

Lemma 2.10. [28, 29] Let us note in passing that *the spinor representation of the linear transformation*

$$\begin{pmatrix} \mathbf{1} & \\ B' & \mathbf{1} \end{pmatrix} \in \mathrm{SO}(\Lambda_{\mathbb{R}}, q)$$

is $\exp[2^{-1}B'\wedge]$ *on the representation space* $H^*(T^{2n};\mathbb{R})$.

2.11. Vertical generalized Hodge structure on $H^*(T^{2n};\mathbb{Q})$: Let $\omega = 2^{-1}\omega_{IJ}dX^I \wedge dX^J$ and $B' = 2^{-1}B'_{IJ}dX^I \wedge dX^J$ be a symplectic form and a real–valued closed 2-form on T^{2n}, respectively. Using ω and B', a linear operator $\mathcal{J}_{B'}$ on $\Lambda_{\mathbb{R}}$ is introduced:

$$\mathcal{J}_{B'} : (\partial_{X^I}, dX^I) \longmapsto (\partial_{X^L}, dX^L) \begin{pmatrix} \delta^L_K & \\ B'_{LK} & \delta_L^K \end{pmatrix} \begin{pmatrix} & (\omega^{-1})^{JK} \\ \omega_{KJ} & \end{pmatrix} \begin{pmatrix} \delta^J_I & \\ -B'_{JI} & \delta_J^I \end{pmatrix}. \tag{2.12}$$

This operator satisfies

$$(\mathcal{J}_{B'})^T \cdot q \cdot \mathcal{J}_{B'} = q, \tag{2.13}$$

so we may find an element $iX_{\omega,B'}$ of the Lie algebra $\mathfrak{so}(\Lambda_{\mathbb{R}}, q)$,

$$\mathcal{J}_{B'} =: \exp\left[\frac{\pi}{2} iX_{\omega,B'}\right], \tag{2.14}$$

and define a representation [28, §8.4]

$$h_{\omega,B'} : S^1 \ni e^{i\alpha} \longmapsto \exp[i\alpha X_{\omega,B'}] \in \mathrm{GL}(\Lambda_{\mathbb{R}}). \tag{2.15}$$

The S^1 subgroup of $\mathrm{SO}(\Lambda_{\mathbb{R}}, q)$ determined this way may be denoted by $S^1_{\omega,B'}$.

We may introduce a representation of the $S^1_{\omega,B'}$ subgroup on $H^*(T^{2n};\mathbb{Q})\otimes\mathbb{R}$ by restricting the spinor representation of $\mathrm{SO}(\Lambda_{\mathbb{R}}, q)$ on $H^*(T^{2n};\mathbb{R})$. This is denoted by $\rho_{\text{spin}}(h_{\omega,B'})$. The

(cf Def. A.8). When the decreasing filtration $W^\bullet$ terminates at $0 \subsetneq W^{2n} \cong \mathbb{Q}$, the set of information to be called a polarization of a generalized Hodge structure will include a bilinear pairing $(-,-)_0 : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to V_{\mathbb{Q}}$ such that $(W^k, W^\ell)_0 \subset W^{k+\ell}$, generalizing the notion of the wedge product on the middle dimensional cohomology group $H^n(M;\mathbb{Q})$ when $\dim_{\mathbb{C}} M = n$. One may also include $(-,-)_{2p} : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to V_{\mathbb{Q}}$ where $(W^k, W^\ell)_{2p} \subset W^{k+\ell+2p}$. A polarization on the Hodge structure on $H^k(M;\mathbb{Q})$ on an abelian variety M with $\dim_{\mathbb{C}} M = n$ has been generalized to $(-,-)_{2(n-k)}$. So, a tentative definition may be to demand a set of information $(-,-)_{2p}$ for $p = n, n-1, \cdots, 0, \cdots, -n$, with a positive definiteness condition similar to the one for a pure rational Hodge structure (see Def. A.8).

representation splits into the representation on $H^{\mathrm{even}}(T^{2n};\mathbb{Q})\otimes\mathbb{R}$ and on $H^{\mathrm{odd}}(T^{2n};\mathbb{Q})\otimes\mathbb{R}$. For a general ω and B', however, we have no reason to expect that there is a filtration structure (there is a sub–representation space defined over $\mathbb{Q}$) like we have in 2.8 or in Def. 2.9.

One can still verify by computation that the vector subspaces [29, §4.1 Ex.2]

$$\mathbb{C}e^{2^{-1}(B'\pm i\omega)}\subset H^*(T^{2n};\mathbb{Q})\otimes\mathbb{C} \tag{2.16}$$

are where the representation $\rho_{\mathrm{spin}}(h_{\omega,B'})$ becomes 1−dimensional, with $\rho_{\mathrm{spin}}(h_{\omega,B'}): e^{i\alpha}\mapsto e^{\mp in\alpha}$. Following some computation, one also finds that the vector spaces

$$\mathbb{C}e^{2^{-1}(B'\pm i\omega)}dX^I\subset H^*(T^{2n};\mathbb{Q})\otimes\mathbb{C} \tag{2.17}$$

for any $I=1,\cdots,2n$ are where the representation becomes 1−dimensional, with $\rho_{\mathrm{spin}}(h_{\omega,B'}):e^{i\alpha}\mapsto e^{\mp i\alpha(n-1)}$. Although it is possible to write down the generators of all the 2^{2n} one dimensional representations in a similar fashion, these are all that we will use in this article.

2.12. The arguments 2.8 and 2.11 are purely mathematical, and are independent of each other. In the context of torus compactification, however, we have a metric G on T^{2n}. When we choose a complex structure I with which G is compatible, we have a natural choice of a symplectic form, the Kähler form $\omega=\omega(-,-):=2^{-1}G(I-,-)$; when we write $\omega=2^{-1}\omega_{IJ}dX^I\wedge dX^J$, then $\omega_{IJ}=I_I^K G_{KJ}=(I^TG)_{IJ}$.

The operators $\mathcal{I}_{B'}$ and $\mathcal{J}_{B'}$ on $\Lambda_{\mathbb{R}}$ commute, and so do $X_{I,B'}$ and $X_{\omega,B'}$ when a common B' is used for both. So, the two U(1) subgroups $S^1_{I,B'}$ and $S^1_{\omega,B'}$ in $\mathrm{SO}(\Lambda_{\mathbb{R}},q)$ commute.

We now proceed to make contact with the following result from string theory.

Lemma 2.13. [23, Prop. 4 and Prop. 8] *Consider the $\mathcal{N}=(1,1)$ SCFT associated with a set of data $(T^{2n};G,B)$. When we specify a pair of rank-n primitive subgroups Γ_f and Γ_b of $H_1(T^{2n};\mathbb{Z})$ so that $\Gamma_f\oplus\Gamma_b\cong H_1(T^{2n};\mathbb{Z})$*, there is a fibration $T^{2n}\to\mathbb{R}^n/\Gamma_b=T^n$. The T–duality transformation along the fiber T^n in string theory implies that there is a lattice isometry

$$g:\Lambda=(\Gamma_f\oplus\Gamma_b)\oplus(\Gamma_f^\vee\oplus\Gamma_b^\vee)\to(\Gamma_f^\circ\oplus\Gamma_b)\oplus((\Gamma_f^\circ)^\vee\oplus\Gamma_b^\vee)=:\Lambda^\circ \tag{2.18}$$

with $g:\Gamma_f\cong(\Gamma_f^\circ)^\vee$ and $g:\Gamma_f^\vee\cong\Gamma_f^\circ$, and there is also an isomorphism f from the total Hilbert space of the $\mathcal{N}=(1,1)$ SCFT for $(T^{2n};G,B)$ to that for $(T^{2n}_\circ;G^\circ,B^\circ)$.

Suppose that I is a complex structure on T^{2n} with which G is compatible. Then I specifies one additional holomorphic (left-mover) U(1) current J_L and one more in right-mover J_R in the superchiral algebra so that the the original $\mathcal{N} = (1,1)$ superconformal algebra extends to an $\mathcal{N} = (2,2)$ superconformal algebra. The T–duality isomorphism f between the Hilbert spaces specifies two current operators $J_L^\circ := fJ_Lf^{-1}$ and $J_R^\circ := -fJ_Rf^{-1}$ in the superchiral algebras of the $\mathcal{N} = (1,1)$ SCFT for $(T_\circ^{2n}; G^\circ, B^\circ)$, but it is not guaranteed that there exists some complex structure I° on $T_\circ^{2n}$ compatible with G° so that the pair J_L° and J_R° is reproduced from I°. We say that $(T^{2n}; G, B; I)$ *has a geometric SYZ-mirror* when such an appropriate complex structure I° exists.

A geometric SYZ–mirror exists for the T–duality along $\Gamma_f \subset H_1(T^{2n}; \mathbb{Z})$ if and only if the following conditions are satisfied:

$$\omega|_{\Gamma_f\otimes\mathbb{R}} = 0, \qquad B|_{\Gamma_f\otimes\mathbb{R}} = 0. \tag{2.19}$$

□

Note that the condition (2.19) does not ask to find an isotropic n-dimensional vector space $\mathbb{R}^n$ within $H_1(T^{2n}; \mathbb{R})$, but to find an isotropic n-dimensional vector space $\mathbb{Q}^n \cong \Gamma_f \otimes \mathbb{Q}$ within $H_1(T^{2n}; \mathbb{Q})$. It is clear that a generic choice of $(G, B; I)$ would not have a geometric SYZ-mirror [28, §9.5].

2.14. Suppose that the $\mathcal{N} = (1,1)$ SCFT for $(T^{2n}; G, B)$ with (J_L, J_R) for a complex structure I has a geometric SYZ–mirror for a T–duality along a rank-n subgroup $\Gamma_f \subset \Gamma_f \oplus \Gamma_b \cong H_1(T^{2n}; \mathbb{Z})$. We use the same notation as in Lemma 2.13.

It is understood in string theory[13] that there is an isomorphism $H^*(T^{2n}; \mathbb{Q}) \cong H^*(T_\circ^{2n}; \mathbb{Q})$ given by the map of D–brane charges having the same physical properties. We abuse the notation and use g for this isomorphism, too. The cohomology group $H^*(T_\circ^{2n}; \mathbb{Q})$ has a grading $\oplus_k H^k(T_\circ^{2n}; \mathbb{Q})$, as well as a filtration structure $W_{h\circ}^\bullet$ (i.e., $W_h^\bullet$ in (2.11) for the mirror theory), and the data (B°, I°) introduces a pure rational Hodge structure (resp. the horizontal generalized rational Hodge structure) on $H^k(T_\circ^{2n}; \mathbb{Q})$ (resp. $H^*(T_\circ^{2n}; \mathbb{Q})$). Those structures can be superimposed on $H^*(T^{2n}; \mathbb{Q})$ by pulling them back via the isomorphism g. Using the

[13] The isometry $g : (\Lambda, q) \cong (\Lambda^\circ, q^\circ)$ induces the isomorphisms $\mathrm{SO}(\Lambda_\mathbb{R}, q) \cong \mathrm{SO}(\Lambda_\mathbb{R}^\circ, q^\circ)$ and $\mathrm{Cliff}(\Lambda_\mathbb{R}, q) \cong \mathrm{Cliff}(\Lambda_\mathbb{R}^\circ, q^\circ)$; we abuse the notation and denote those two isomorphism as g. Thus, the left-$\mathrm{Cliff}(\Lambda_\mathbb{R}^\circ, q^\circ)$-module $\mathfrak{a}_{\Gamma_f^\circ+\Gamma_b}$ can be regarded as a left-$\mathrm{Cliff}(\Lambda_\mathbb{R}, q)$ module as well. A linear map $g : \mathfrak{a}_{\Gamma_f+\Gamma_b} \to \mathfrak{a}_{\Gamma_f^\circ+\Gamma_b}$ is determined by demanding that it is compatible with the action of $\mathrm{Cliff}(\Lambda_\mathbb{R}, q)$. Combining this isomorphism with the cohomology interpretation in footnote 10, we obtain $g : H^*(T^{2n}; \mathbb{Q}) \to H^*(T_\circ^{2n}; \mathbb{Q})$ in the main text (e.g., [28, §3.3, §3.5 and §9.3]).

isometry $g : (\Lambda, q) \to (\Lambda^\circ, q^\circ)$, $g^{-1}\mathcal{I}_{B^\circ} g = \mathcal{J}_B$ and $g^{-1} S^1_{I^\circ, B^\circ} g = S^1_{\omega, B}$ [20]. In the spinor representation, the horizontal generalized Hodge structure $(\rho_{\rm spin}(h_{I^\circ,B^\circ}), W^\bullet_{h\circ})$ on $H^*(T^{2n}_\circ;\mathbb{Q})$ is mapped into a generalized rational Hodge structure on $H^*(T^{2n};\mathbb{Q})$ by $(\rho_{\rm spin}(h_{\omega,B}), g^*(W^\bullet_{h\circ}))$. We call this generalized rational Hodge structure as the *vertical generalized rational Hodge structure.*[14] See Fig. 1 for an illustration.

The rational Hodge structure on $H^n(T^{2n}_\circ;\mathbb{Q})$ by I° is polarized with respect to the wedge product on $T^{2n}_\circ$:

$$(W^n_{h\circ}/W^{n+2}_{h\circ}) \times (W^n_{h\circ}/W^{n+2}_{h\circ}) \ni (\psi, \chi) \longmapsto \int_{T^{2n}_\circ} \psi \wedge \chi \in \mathbb{Q}. \tag{2.21}$$

When this bilinear form (symmetric if n is even, and anti–symmetric if n is odd) is pulled back by g to $H^*(T^{2n};\mathbb{Q})$, it becomes

$$\begin{aligned} g^*(W^n_{h\circ}/W^{n+2}_{h\circ}) \ &\times g^*(W^n_{h\circ}/W^{n+2}_{h\circ}) \ni (g^*(\psi), g^*(\chi)) \\ &\longmapsto (-1)^{\frac{n(n-1)}{2}} \int_{T^{2n}} \left(\sum_{k=0}^{n} (-1)^k \Pi_{2k} g^*(\psi) \right) \wedge g^*(\psi) \in \mathbb{Q}, \end{aligned} \tag{2.22}$$

where Π_{2k} is the projection $H^*(T^{2n};\mathbb{Q}) \to H^{2k}(T^{2n};\mathbb{Q})$.

Remark 2.15. The most natural choice of the operators $\mathcal{I}_{B'}$ and $\mathcal{J}_{B'}$ are for B' equal to the B–field in the data $(T^{2n}; G, B)$ of an $\mathcal{N} = (1,1)$ SCFT. In this case, $\mathcal{I}_B\mathcal{J}_B$ is an operation multiplying -1 to the right-moving momentum on $\Lambda_\mathbb{R}$ [23, eq. (2.10)]. It is still possible, mathematically, to define those operators with B' chosen differently than B itself; then different U(1) subgroups $S^1_{I,B'}$ and $S^1_{\omega,B'}$ are specified within $\mathrm{SO}(\Lambda_\mathbb{R}, q)$. For a technical reason in the presentation, we will also use such subgroups in footnote 34.

2.16. Suppose that there are two geometric SYZ-mirrors for the $\mathcal{N} = (1,1)$ SCFT with a set of data $(T^{2n}; G, B; I)$. Let $\Gamma_{fi} \oplus \Gamma_{bi} \cong H_1(T^{2n};\mathbb{Z})$ for $i = 1, 2$ be the split of the 1-cycles into those in the fiber (to be taken T-dual) and those in the base, and g_i and f_i (with $i = 1, 2$) the corresponding lattice isometries and the Hilbert space isomorphisms, respectively.

[14] The vertical generalized rational Hodge structure splits into pure rational Hodge structures of weights ranging from 0 to $2n$, when the condition

$$\omega|_{\Gamma_b \otimes \mathbb{R}} = 0, \qquad B|_{\Gamma_b \otimes \mathbb{R}} = 0 \tag{2.20}$$

is also satisfied.

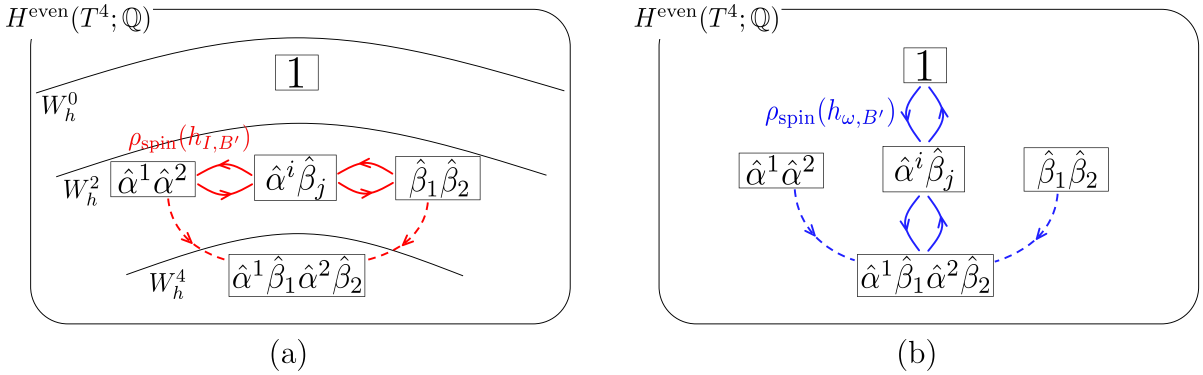

Figure 1: This figure illustrates how the (a) horizontal and (b) vertical S^1 subgroups act on $H^*(T^4;\mathbb{R})$, and how the filtration $W_h^\bullet$ is introduced on $H^*(T^4;\mathbb{Q})$. See also Fig. 2 for the filtrations $g^*(W_{h\circ}^\bullet)$.

Both representations $\rho_{\text{spin}}(h_{I^\circ_{(i)},B^\circ_i})$ on $H^*(T^{2n}_{\circ(i)};\mathbb{R})$ for $i=1,2$ are pulled back by g_i to one identical representation on $H^*(T^{2n};\mathbb{R})$, and that is the representation $\rho_{\text{spin}}(h_{\omega,B})$. It is therefore economical to deal with $\rho_{\text{spin}}(h_{\omega,B})$ instead of $\rho_{\text{spin}}(h_{I^\circ_{(i)},B^\circ_i})$. The gradings and the filtrations pulled back to $H^*(T^{2n};\mathbb{Q})$—$g_i^*(H^k(T^{2n}_{\circ(i)};\mathbb{Q}))$ and $g_i^*(W^k_{h\circ(i)})$—are however not identical[15] for different geometric SYZ-mirrors $i=1,2$ (cf Fig. 2).

2.5 Coarse Classification of CM-type Abelian Surfaces

A complex torus $M=\mathbb{C}^n/\mathbb{Z}^{\oplus n}=(T^{2n};I)$ of dimension n is regarded as an *abelian variety* when there exists a *polarization*, which means the existence of a $\psi\in H^2(M;\mathbb{Z})\cap H^{1,1}(M;\mathbb{R})$ such that the bilinear form $\psi(I-,-):(X,Y)\mapsto\psi(IX,Y)$ for $X,Y\in H_1(T^{2n};\mathbb{R})$ is positive definite. It is a non-trivial condition on I whether a polarization exists or not. Mathematicians tend to favor abelian varieties over general complex tori because abelian varieties may be treated as algebraic varieties (rather than complex analytic manifolds). String theorists, however, do not have any a priori reason to be in favor of a complex structure on T^{2n} that allows a polarization over those that do not. So, here, we introduce the following definition for a general complex torus that is not necessarily an abelian variety.

Definition 2.17. Let $M=(T^{2n};I)$ be a complex torus of dimension n; then a rational Hodge structure is given on $H^1(M;\mathbb{Q})$. The following two conditions are known to be equivalent:[16]

[15]The pulled back filtrations, $g_i^*(W^\bullet_{h\circ(i)})$, are identical, if $\Gamma_{f1}=\Gamma_{f2}$ (even when $\Gamma_{b1}\neq\Gamma_{b2}$).

[16]The proof of Props. 17.3.4 and 17.3.5 of [33] does not assume that the Hodge structure in question admits

(i) The algebra $\mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ over $\mathbb{Q}$ contains a commutative semi-simple subalgebra[17] of dimension $2n$.

(ii) The Hodge group of the Hodge structure $\mathrm{Hg}(M)$ is commutative.

The definition of a Hodge group is given in Def. A.9; very little intuition on the Hodge group is required, however, in following the related arguments in section 3.1.

When either one of the above conditions (and hence both) are satisfied, we say that the complex torus has/is with *sufficiently many complex multiplications*, and also that the rational Hodge structure is *with sufficiently many CM*. In the case a complex torus M (and its rational Hodge structure on $H^1(M;\mathbb{Q})$) with the property (i), (ii) admits a polarization, we say that M is a *CM abelian variety*, and the Hodge structure is of *CM-type*.

Remark 2.18. It may seem a little odd to use different jargons for one and the same properties, (i) and (ii), depending on whether existence of a polarization is guaranteed or not. Such a choice of jargons partially reflects the fact that the properties (i) and (ii) mean a lot more when they are combined with a polarization.

---

a polarization.

[17] When a finite dimensional semi-simple algebra over $\mathbb{Q}$ is commutative, then it is of the form of $\oplus_{\alpha\in\mathcal{A}}F_\alpha$, the direct sum of a finite number of number fields F_α. Conversely, an algebra of this form is always semi-simple and commutative.

An addendum in arXiv ver. 2: only one of the authors (TW) is held responsible for the rest of this footnote. Although the condition (i) is a natural generalization of the definition of CM-type from abelian varieties to complex tori, there is another way to generalize that is equally natural. One might also think of defining complex tori with *sufficiently many complex multiplications* by imposing the following condition: (i′) The semi-simplification of the algebra $\mathfrak{R} := \mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ over $\mathbb{Q}$, i.e., the quotient $\mathfrak{R}/J$ by the radical $J := J(\mathfrak{R})$ of $\mathfrak{R}$, contains a commutative semi-simple subalgebra of dimension $2n$. It turns out (as explained shortly), however, that the two conditions (i) and (i’) are equivalent; so it does not matter which one is used for the definition.

To see that the condition (i) implies the condition (i′), one just has to note that a semi-simple subalgebra $F \subset \mathfrak{R}$ is injectively mapped into the quotient $\mathfrak{R}/J$. To see that the condition (i′) implies (i), think of a case where M is an indecomposable complex torus of dimension n for simplicity (we use terminology of [34]). The algebra $\mathfrak{R}$ is a local algebra, and the division algebra $\mathfrak{R}/J$ contains a commutative algebra F (a field F in this simple case) because of the condition (i′). Now, there is a sequence of complex subtori $M \supset JM \supset J^2M \supset \cdots \supset J^mM = \{0\}$; here, we adapt the idea of Loewy series of a module ([35, p. 346], [36, §1]) to a complex torus. The division algebra $\mathfrak{R}/J$ and its subfield F is realized on each of the complex tori $J^iM/J^{i+1}M$ with $i = 0, \cdots, m-1$ as the endomorphism algebra. If a non-zero element $\phi + J$ of $\mathfrak{R}/J$ or F were to be realized trivially in any one of the tori $J^iM/J^{i+1}M$, then $\phi \in \mathfrak{R}$ must be nilpotent, which is a contradiction. So, a degree $2n$ field F has an embedding into $\mathrm{End}_{\mathbb{Q}}(J^iM/J^{i+1}M)$ for some $i \in \{0, \cdots, m-1\}$, and we learn that the complex torus $J^iM/J^{i+1}M$ is of n dimension, which further implies that $J^{i+1}M = 0$, and $J^iM = M$, so $m = 1$, $i = 0$, and $J(\mathfrak{R}) = 0$ in the end. Under the condition (i′), the semi-simple commutative subalgebra $F \subset \mathfrak{R}/J$ is actually a subalgebra of $\mathfrak{R}$ (i.e., the condition (i)), and an indecomposable complex torus satisfying the condition (i′) is always a simple complex torus. □

Here are a few properties that hold true only when a polarization exists (see e.g., [33, 37, 6]):

- The algebra $\mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ is semi-simple, so this algebra has the structure (A.5).
- The semi-simple algebra $\mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ with the structure (A.5) acts faithfully[18] on the vector space $H^1(M;\mathbb{Q})$ (by definition); when this algebra contains a $2n$-dimensional commutative subalgebra (so, its structure is of the form in footnote 17), that means that the division algebra D_α is commutative, (i.e., $D_\alpha = k_\alpha$ and $q_\alpha = 1$), and the number field F_α is a degree-n_α extension[19] of the number field k_α.
- Both k_α and F_α are CM-fields.

A little more information is provided in section 3.1 on complex tori with sufficiently many complex multiplications. Although complex tori with sufficiently many complex multiplications are more general than CM abelian varieties (making it desirable to have a theory relating rational SCFTs with such tori), we will be able to confirm such a connection only for CM abelian varieties. This is for scientific reasons, not a matter of mathematical taste, preference or interests.

For that reason, it makes sense to prepare ourselves to work specifically with CM abelian varieties. Let us quote a result of classification of CM-type abelian varieties of $n = 2$ dimensions. That is essentially done by classifying the CM algebras $\oplus_{\alpha\in\mathcal{A}} F_\alpha$ of dimension $2n = 4$.

Lemma 2.19. [6, pp.64–65 Ex.8.4.(2)] *There are four different kinds of CM algebras $\oplus_\alpha F_\alpha$ over $\mathbb{Q}$ of dimension 4.*

(A) The CM algebra is a CM field (i.e., $|\mathcal{A}| = 1$), and $F = F_\alpha \cong \mathbb{Q}[x, y]/(y^2 - d, x^2 - p)$ for some square-free integers $d > 1$ and $p < 0$. This field F is an extension of the imaginary quadratic field $K^{(2)} \cong \mathbb{Q}[x]/(x^2 - p)$. The totally real subfield of F is $\mathbb{Q}[y]/(y^2 - d)$.

(A') The CM algebra is of the form $K_1^{(2)} \oplus K_2^{(2)}$, where $K_1^{(2)} \cong \mathbb{Q}[x_1]/(x_1^2 - p_1)$ and $K_2^{(2)} \cong \mathbb{Q}[x_2]/(x_2^2 - p_2)$ are imaginary quadratic fields that are not mutually isomorphic. That is, p_1, p_2 are negative square-free integers, and $p_1 \notin p_2(\mathbb{Q}^\times)^2$. $|\mathcal{A}| = 2$ in this case.

[18] A structure theory on $\mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ of a complex torus, not necessarily with a polarization, is found in [34, §1.7 and §1.8].

[19] Footnote 21 provides a pedagogical explanation on how to construct an extension F_α/k_α.

(B, C) The CM algebra is a CM field (i.e., $|\mathcal{A}| = 1$), $F = F_\alpha =: K$, that does not contain a CM subfield. Such a degree-4 CM field always has a structure $K \cong \mathbb{Q}[x,y]/(y^2-d, x^2-p-qy)$ for some square-free integer $d > 1$ and rational numbers p, q such that $p < 0$, $q \neq 0$ and $d' := p^2 - q^2 d > 0$. The two distinct cases $d' \in d(\mathbb{Q}^\times)^2$ and $d' \notin d(\mathbb{Q}^\times)^2$ are called the case (B) and (C), respectively; the field extension $K/\mathbb{Q}$ is Galois and non-Galois, respectively, in the two cases. The totally real subfield is $\mathbb{Q}[y]/(y^2 - d)$.

The distinction between the cases B and C is not very important in the analysis in this article.

The case (A) looks as if it is the cases (B, C) with just setting $q = 0$. There is a clear difference between the case (A) and the cases (B, C), however. The difference is seen in the reflex field of the CM field F in the case (A) and the field K in the cases (B, C) (e.g., [6, 37]). To be more explicit:

2.20. For the CM field F in **case (A)**, the four embeddings $F \hookrightarrow \overline{\mathbb{Q}}$ are denoted by $\tau_{\pm\pm}$, where $\tau_{\pm *} : y \mapsto \pm\sqrt{d}$ and $\tau_{*\pm} : x \mapsto \pm\sqrt{p} = \pm i\sqrt{-p}$. Throughout in this article, we mean by $\sqrt{d}$ for $d \in \mathbb{R}_{>0}$ the positive square root, and by $\sqrt{p} = i\sqrt{-p}$ for $p \in \mathbb{R}_{<0}$ the square root in the upper half complex plane.

There seem to be two choices of a CM-type of the CM-field, $\{\tau_{++}, \tau_{-+}\}$ and $\{\tau_{++}, \tau_{--}\}$; in fact, we have an alternative presentation $F \cong \mathbb{Q}[x', y]/(y^2 - d, (x')^2 - pd)$ due to the isomorphism $xy \leftrightarrow x'$, and the set of embeddings $\{\tau_{++}, \tau_{--}\}$ for $F \cong \mathbb{Q}[x,y]/(y^2 - d, x^2 - p)$ is regarded as $\{\tau_{++}, \tau_{-+}\}$ for $F \cong \mathbb{Q}[x', y]/(y^2 - d, (x')^2 - pd)$. So, we do not lose generality by thinking only of the CM-type $\Phi := \{\tau_{++}, \tau_{-+}\}$.

For the CM-type (F, Φ), the reflex field F^r is $\mathbb{Q}[\xi^r]/((\xi^r)^2 - p)$, which is an imaginary quadratic field. The reflex field of the reflex field is $F^{rr} = \mathbb{Q}(\sqrt{p})$, and F^{rr} is a proper-subfield of F in the case (A). It is also easy to see this directly from the fact that the CM-type Φ is not primitive, but is induced from the CM-type $(K^{(2)}, \tau_{*+} : x \mapsto \sqrt{p})$ [6, §8].

For the CM field K in the **cases (B, C)**, on the other hand, the four embeddings $K \hookrightarrow \overline{\mathbb{Q}}$ are denoted by $\tau_{\pm\pm}$, where

$$\tau_{\pm *} : y \mapsto \pm\sqrt{d}, \qquad \tau_{\pm +} : x \mapsto \sqrt{p \pm q\sqrt{d}} = i\sqrt{-p \mp q\sqrt{d}}, \tag{2.23}$$

$$\tau_{\pm -} : x \mapsto -\sqrt{p \pm q\sqrt{d}} = -i\sqrt{-p \mp q\sqrt{d}}. \tag{2.24}$$

We introduce a short-hand notation $\sqrt{+} := \sqrt{p + q\sqrt{d}}$ and $\sqrt{-} := \sqrt{p - q\sqrt{d}}$ for the pure imaginary complex numbers in the upper half plane and use it for the sake of compactness of notation in this article.

There seem to be two inequivalent choices of a CM-type of the CM-field K, namely, $\{\tau_{++}, \tau_{-+}\}$ and $\{\tau_{++}, \tau_{--}\}$. We can change the presentation of the field K (with different values of p, q) so that the choice $\{\tau_{++}, \tau_{--}\}$ is regarded as $\{\tau_{++}, \tau_{-+}\}$ in the new presentation, as we have done in the case (A). So, we do not lose generality by considering only one CM-type $\Phi := \{\tau_{++}, \tau_{-+}\}$.

The reflex field K^r of (K, Φ) is not (necessarily) isomorphic to K in the cases (B, C), but is a degree-4 number field

$$K^r \cong \mathbb{Q}[y', \xi^r]/((y')^2 - d', (\xi^r)^2 - 2p + 2y'). \tag{2.25}$$

The reflex field of K^r, denoted by K^{rr}, is K itself in the cases (B, C). That is the difference between the case at hand and case (A); this difference in algebraic notation/terminology is also reflected in geometric notation/terminology concerning abelian varieties as we quote a statement below (Lemma 2.22).

Before moving on, however, let us introduce notations $\tau^r_{\pm\pm}$ for the four embeddings of the reflex field $K^r \hookrightarrow \overline{\mathbb{Q}}$.

$$\tau^r_{\pm *} : y' \mapsto \pm\sqrt{d'}, \qquad \tau^r_{\pm +} : \xi^r \mapsto \sqrt{p + q\sqrt{d}} \pm \sqrt{p - q\sqrt{d}}, \tag{2.26}$$

$$\tau^r_{\pm -} : \xi^r \mapsto -\left(\sqrt{p + q\sqrt{d}} \pm \sqrt{p - q\sqrt{d}}\right). \tag{2.27}$$

2.21. Let us also write down a little bit of information on the reflex field in the **case (A')**, because we use that later in this article. The two embeddings of the imaginary quadratic fields $K_i^{(2)}$ are given by $\tau_{i\epsilon_i} : x_i \mapsto \epsilon_i\sqrt{p_i} = \epsilon_i i\sqrt{-p_i}$ for $\epsilon_i \in \{\pm\}$. A CM-type[20] of the CM algebra $K_1^{(2)} \oplus K_2^{(2)}$ must be for $\Phi = \{(\tau_{1\epsilon_1}, \tau_{2\epsilon_2})\}$ for some choice of (ϵ_1, ϵ_2). For any one of them, the reflex field is $K^r = \mathbb{Q}[x_1, x_2]/(x_1^2 - p_1, x_2^2 - p_2)$, which is isomorphic to $\mathbb{Q}[\xi^r, y']/((\xi^r)^2 - p_1, (y')^2 - d')$ with $d' = p_1p_2 > 0$ through $x_1 \mapsto \xi^r$ and $x_1x_2 \mapsto y'$. The four embeddings may be denoted by $\tau^r_{\epsilon', \epsilon^r}$, where $\tau^r_{\pm *} : y' \mapsto \pm\sqrt{d'}$ and $\tau^r_{*\pm} : \xi^r \mapsto \pm\sqrt{p_1}$. $\square$

Let us now quote the known result stating how the classification of degree-4 CM algebras above is translated to the classification of CM-type abelian varieties of complex dimension 2 (abelian surfaces).

Lemma 2.22. *Let M be an abelian surface of CM-type. Then it must be in one of the following mutually exclusive cases.*

[20] See [37, Def. 1.17] for the definition of the reflex field of a CM algebra that is not a CM field.

(A) There is an isogeny $\varphi : M \longrightarrow E \times E$ where E is an elliptic curve of CM-type with $\mathrm{End}(H^1(E;\mathbb{Q}))^{\mathrm{Hdg}} \cong K^{(2)} \cong \mathbb{Q}[x]/(x^2-p)$. In this case,

$$\mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}} \cong M_2(K^{(2)}); \tag{2.28}$$

for any square-free integer $d > 1$, one can find within[21] the algebra $M_2(K^{(2)})$ a subfield F of the property (A) in the classification in Lemma 2.19.

(A') There is an isogeny $\varphi : M \longrightarrow E_1 \times E_2$ where E_i is an elliptic curve of CM-type with $\mathrm{End}(H^1(E_i;\mathbb{Q}))^{\mathrm{Hdg}} \cong K_i^{(2)} \cong \mathbb{Q}[x]/(x^2-p_i)$ (for both $i = 1,2$).

(B,C) M does not contain an abelian subvariety. $\mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}} \cong K$.

Abstract elements denoted by x, y are then regarded as 'Hodge structure preserving' endomorphisms on $H^1(M;\mathbb{Q})$.

3 Choice of Complex Structure

As we have remarked in Discussion 2.4, there is no way not choosing a complex structure on T^{2n} when we wish to establish a Gukov–Vafa-like characterization of rational T^{2n}-target (S)CFTs. On one hand, it is desirable to find a characterization statement that works well for a broader class of complex structures on T^{2n}. For example, it may be natural for algebraic geometers to pay attention only to a complex structure I such that $(T^{2n}; I)$ is an abelian

[21] Such a CM field $F \cong \mathbb{Q}[x,y]/(x^2-p, y^2-d)$ within $M_2(K^{(2)})$ can be constructed as follows. First, think of a map $\phi_x : z^1 \mapsto \sqrt{p}z^1$, $z^2 \mapsto \sqrt{p}z^2$. The pull-back ϕ_x^* generates the center $K^{(2)}\mathbf{1}_{2\times 2}$ of the algebra $M_2(K^{(2)})$. Next, for a square-free integer $d > 1$ and a complex multiplication $\xi \in \mathrm{End}(H_1(E;\mathbb{Q}))^{\mathrm{Hdg}}\backslash\{0\}$, think of a map $\phi_{d,\xi} : z^1 \mapsto d\xi z^2$, $z^2 \mapsto \xi^{-1}z^1$. Then $(\phi_{d,\xi}^*) \in \mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ has the property of the generator y, so we may set F to be the subalgebra of $M_2(K^{(2)})$ generated by $x = \phi_x^*$ and $y = \phi_{d,\xi}^*$.

Such a subfield F within the algebra $M_2(K^{(2)})$ has the following property characterized by a polarization $\mathcal{Q}_{d,\xi} \in \sqrt{p}\left(dz^1 \wedge d\bar{z}^{\bar{1}} + d\mathrm{Nm}(\xi)dz^2 \wedge d\bar{z}^{\bar{2}}\right)\mathbb{Q}$. When we assign to $\phi \in \mathrm{End}(H_1(M;\mathbb{Q}))^{\mathrm{Hdg}} \cong M_2(K^{(2)})$ another endomorphism $\phi' \in \mathrm{End}(H_1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ by $\mathcal{Q}(\phi'-,-) = \mathcal{Q}(-,\phi-)$ (this is called *Rosati involution* with respect to $\mathcal{Q}$), then $\phi'_x = -\phi_x$, and $\phi'_{d,\xi} = \phi_{d,\xi}$, so $\phi' \in F$ for any $\phi \in F$ (i.e., the subfield F for d, ξ is closed under the involution by $\mathcal{Q}_{d,\xi}$). See [37, Prop. 3.6 (b)] for more information.

All above in this footnote are written by mathematicians in many articles explaining the theory of complex multiplication of abelian varieties, but are often expressed only in abstract and general terms. So, we pursued a hands-on style of presentation preferred by string theorists here.

In the context of string theory, it is usually not motivated to fix an embedding of a complex analytic manifold M to a projective space to see it as an algebraic variety. String theorists are not worried about too many automorphisms either. So, there is no particular reason to restrict one's attention only to a proper subalgebra F of $M_2(K^{(2)})$. It is still good to know that $M_2(K^{(2)})$ contains a subfield F that is CM and a degree-2 extension of $K^{(2)}$, because that is enough to be able to apply a useful fact written as Lemma A.11.

variety (e.g. [19]), i.e., an object in the category of algebraic varieties (instead of a general complex torus as an object of the category of complex analytic manifolds). As string theorists, however, we should give a thought whether such a characterization has a chance to work for a complex structure not necessarily with a polarization. That is the subject of section 3.1.

On the other hand, if there is a choice of a complex structure I that is motivated well in string theory, there is a chance that we have a sharper/clearer characterization statement for rational T^{2n}-target (S)CFTs. That is what we aim for in section 3.2.

3.1 Polarization

Let us first extract more information from the definition of a complex torus M with sufficiently many complex multiplications.

3.1. As a direct consequence of the definition, there must be an algebra of endomorphisms of the form $\oplus_{\alpha\in\mathcal{A}}F_\alpha$ acting faithfully on $H^1(M;\mathbb{Q})$; here, F_α is a number field. Since the action is faithful, the comparison of the dimensions implies that the vector space $H^1(M;\mathbb{Q})$ should have a structure $H^1(M;\mathbb{Q}) \cong \oplus_{\alpha\in\mathcal{A}}[H^1(M;\mathbb{Q})]_\alpha$, where $[H^1(M;\mathbb{Q})]_\alpha$ is a 1-dimensional vector space of F_α. See footnote 17 and Lemma A.10 for more background information. We can apply the following discussion to individual pairs F_α and $[H^1(M;\mathbb{Q})]_\alpha$, so we drop the subscript now.

Any endomorphism in $F \subset \mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ maps the Hodge (1,0) and (0,1) components of $H^1(M;\mathbb{Q})$ to themselves. So, the action of the endomorphisms in F can be diagonalized simultaneously on the two Hodge components separately. The simultaneous eigenstates of the action of F explained in Lemma A.11 should therefore belong either to the (1,0) component, or to the (0,1) component. The set $\mathrm{Hom}(F,\mathbb{C})$ of embeddings of the field F is therefore separated into two subsets, $\Phi \subset \mathrm{Hom}(F,\mathbb{C})$ for the (1,0) components and $\overline{\Phi}$ for the (0,1) components. Moreover, the set $\overline{\Phi}$ consists of the embeddings in Φ followed by the complex conjugation in $\mathbb{C}$. This means that a number field F in the context of a complex torus with sufficiently many complex multiplications must be a totally imaginary field.

Let F be a totally imaginary field of degree $2n$, and $\Phi = \{\tau_a\} \subset \mathrm{Hom}(F,\mathbb{C})$ a set of n embeddings, any two of which are not mutually complex conjugate of the other. Then one can construct a complex torus $\mathbb{C}^n/\mathbb{Z}^{\oplus 2n}$ of n-dimensions by choosing a basis $\{\eta_{I=1,\cdots,2n}\}$ of $F/\mathbb{Q}$ and setting $\mathbb{Z}^{\oplus 2n} \hookrightarrow \mathbb{C}^n$ to be $(n_1, n_2, \cdots, n_{2n}) \mapsto (\tau_1(n_I\eta_I),\ \tau_2(n_I\eta_I), \cdots, \tau_n(n_I\eta_I)) \in \mathbb{C}^n$. The $2n$ vectors $(\tau_{a=1,\cdots,n}(\eta_I)) \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ for $I = 1, \cdots, 2n$ are automatically linearly independent over $\mathbb{R}$; to see this, it is enough to note that the $2n\times 2n$ matrix $(\tau_a(\eta_I), \bar{\tau}_a(\eta_I))_{I,a\bar{a}}$

has a non-zero determinant (see Lemma A.11). The algebra of endomorphisms of this complex torus contains a subalgebra isomorphic to F (use Lemma A.12).

Remark 3.2. CM fields constitute a special subclass of totally imaginary fields. A complex torus with sufficiently many complex multiplications is a CM abelian variety if and only if its endomorphism algebra $\oplus_\alpha F_\alpha$ is made of totally imaginary fields F_α that are all CM fields. The implication $\Rightarrow$ is from the 3rd property quoted in Rmk. 2.18. The implication $\Leftarrow$ is from Lemma 4.1. Examples of totally imaginary fields that are not CM fields are found[22] in the database LMFDB (`www.lmfdb.org`). For example, $F = \mathbb{Q}[x]/(x^4 - 2x^2 + 2)$.

Having developed intuitions[23] on complex tori with sufficiently many complex multiplications, however, let us see

Proposition 3.3. [19, Thm. 2.5] *Let $M = (T^{2n}; I)$ be an abelian variety, i.e., a complex torus that admits a polarization. If there exists a constant metric G compatible with I that is rational in the sense of (2.4), then the polarized rational Hodge structure on $H^1(M;\mathbb{Q})$ is of CM-type.*

For a set of data $(T^{2n}; G, B)$ for which the (S)CFT is rational, there is always a complex structure I with which G is compatible and which admits a polarization (see Cor. 2.7 and the discussion that follows). So, Prop. 3.3 above is not an empty statement for *any* T^{2n}-target rational (S)CFTs.

The proof of [19, Thm. 2.5 + Prop. 2.4] is just as informative as the statement itself. Prop. 2.4 of [19] proves that the properties (i) and (ii) in Def. 2.17 are equivalent to the property

(iii) the Hodge group $\mathrm{Hg}(M)(\mathbb{R})$ is compact

for an abelian variety M; Thm. 2.5 of [19] proves the compactness (iii) of $\mathrm{Hg}(M)(\mathbb{R})$ when there exists a rational G that is compatible with I, and hence the properties (i) and (ii) in

[22] Math StackExchange entry "totally imaginary number field of degree 4"
`https://math.stackexchange.com/questions/4372232/`

[23] In the case of CM abelian varieties, the notion of a primitive CM-type $(K_\alpha, \Phi_\alpha^{rr})$ is available [6], so one can work out the embedding of the algebra $\oplus_\alpha F_\alpha$ into the entire endomorphism algebra $\mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ is determined from the CM type (F_α, Φ_α). For a general complex torus with sufficiently many complex multiplications, the authors have not made enough effort to come up with an alternative to the primitivity of CM type; so the authors are not ready to write down a statement similar to the 2nd item in Rmk. 2.18 in connection with the theory of structure of $\mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ for a general complex tori in [34, §1.7 and §1.8].

Def. 2.17. In proving the equivalence between the properties (iii) and (i, ii), however, Ref. [19] uses the fact that $\mathrm{Hg}(M)(\mathbb{R})^{\mathrm{Ad}(h(i))}$ is compact; to prove the compactness of this group, Thm. 1.3.16 of [38][24] uses the positive definiteness of a polarization of the rational Hodge structure. To conclude, the equivalence between the properties (iii) and (i, ii) breaks down when the rational Hodge structure $H^1(M;\mathbb{Q})$ does not necessarily have a polarization.

In our context, even when there is a constant rational metric G compatible with the complex structure I of a complex torus $M = (T^{2n}; I)$, we cannot derive the property (i), the presence of sufficiently many complex multiplications (endomorphisms), if I is not polarized. We could pay attention to complex tori $M = (T^{2n}; I)$ satisfying the property (i) in Def. 2.17, but it is not obvious whether there exists a constant rational metric compatible with the complex structure I (Discussion 4.4 constructs rational metrics satisfying (2.4), by exploiting properties of CM fields not available to a general totally imaginary field). For this reason, we pay attention only to complex structures I that admit polarization in the rest of this article.

There are countably infinitely many such complex structures I for a constant rational metric G on T^{2n}; complex structures compatible with G are parametrized by S^2 [39], and once a 2-form ψ with $\int \psi \wedge \psi > 0$ is chosen from $H^2(T^4;\mathbb{Q})$, then we should choose the direction of the Kähler form ω for the metric G in the way $\mathbb{R}\omega$ includes the projection of ψ to Π_G in the notation to be used in section 3.2, so that ψ becomes Hodge (1,1) type (cf the discussion in between (3.3) and (3.4)). There are countably infinitely many choices of such ψ, and hence of a polarizable complex structure I. Existence of countably infinitely many complex structures is also understood naturally from the way Cor. 2.7 is proven in [17].

3.2 Transcendental Part of the B-field

We may deal with all the polarizable complex structures I on T^{2n} with which a given metric G is compatible and try to characterize the rational Hodge structures when $(T^{2n}; G, B)$ yields a rational (S)CFT. That is done in section 5.2.3. It is also an option to impose further conditions on the choice of I and try to characterize the Hodge structures for rational (S)CFTs for such a smaller class of complex structures. That is what we do in Thms. 5.5, 5.7 and 5.8, built on Thm. 4.6. For them to make sense, however, we should prove the following

Proposition 3.4. This is for the case $n = 2$. *Let $(T^{2n=4}; G, B)$ be a set of data for which the (S)CFT is rational. Then there exists a polarizable complex structure I on T^4 with which G*

[24] We refer to the LNM version, not to its arXiv versions.

is compatible, and the B-field only has the Hodge $(1,1)$ *component with respect to that* I*. In particular, the B-field is in algebraic part* $\mathcal{H}^2(T^4_I)$.

Here,

Definition 3.5. For a general Kähler manifold M of dimension n,

$$\mathcal{H}^2(M) := H^{1,1}(M;\mathbb{R}) \cap H^2(M;\mathbb{Q}) \tag{3.1}$$

is said to be the *algebraic part* of $H^2(M;\mathbb{Q})$. When $n=2$, the orthogonal complement

$$T_M \otimes \mathbb{Q} := \left[\mathcal{H}^2(M;\mathbb{Q})^{\perp} \subset H^2(M;\mathbb{Q})\right] \tag{3.2}$$

with respect to the wedge product is said to be the *transcendental part*. The rational Hodge structure on $H^2(M;\mathbb{Q})$ given by I has a decomposition into the substructures on $\mathcal{H}^2(M)$ and $T_M \otimes \mathbb{Q}$; the substructure on $\mathcal{H}^2(M)$ is of level-0 and that on $T_M \otimes \mathbb{Q}$ of level-2. For a 2-form $\psi \in H^2(M;\mathbb{R})$, its decomposition into $\mathcal{H}^2(M)\otimes\mathbb{R} \oplus T_M \otimes \mathbb{R}$ is denoted by $\psi^{\rm alg} + \psi^{\rm transc}$, and are called the *algebraic and transcendental parts/components*.

Proof of Prop. 3.4: Recall that the metric G determines the real 3-dimensional vector subspace Π_G of $H^2(T^4;\mathbb{R})$ that consists of 2-forms that are self-dual under the Hodge-* operation with respect to the metric G. Choice of a complex structure I compatible with G is to specify one direction for $\omega = 2^{-1}G(I-,-)$ within Π_G; so, the choice of I comes with a variety S^2 [39, §2]; the two directions in Π_G orthogonal to ω with respect to the wedge product supports the holomorphic (2,0) form Ω_M on T^4. Recall also that any 2-form can be decomposed into the self-dual component and the anti-self-dual component under the Hodge-* operation; let $B = B_\parallel + B_\perp$ be the decomposition of the B-field.

When $B_\parallel = 0$, automatically there is no Hodge (2,0) or (0,2) component in $B = B_\perp$, regardless of which direction in Π_G is chosen (and of how complex structure I is chosen). We just have to choose any I in S^2 such that a polarization exists (such an I exists; we have already seen that at the end of section 2.3 for a rational G).

When $B_\parallel \neq 0$, there is virtually no free choice for I after requiring that the Hodge (2,0) component is absent; we have to choose $\omega \in \mathbb{R}B_\parallel$. Choosing $\omega \in \mathbb{R}_{<0}B_\parallel$ instead of $\omega \in \mathbb{R}_{>0}B_\parallel$ is nothing more than declaring holomorphic coordinates on T^4 as anti-holomorphic coordinates instead. So, we fix $\omega = 2^{-1}G(I-,-)$ by the condition $\omega \in \mathbb{R}_{>0}B_\parallel$, and prove that there is a polarization under I.

To this end, note that $\int \Omega_M \wedge B_{\parallel} = 0$ and $\int \Omega_M \wedge B_{\perp} = 0$, which is equivalent to

$$\int_{T^4} \Omega_M \wedge B = 0, \qquad \int_{T^4} \Omega_M \wedge (*B) = 0. \tag{3.3}$$

So, both B and $*B$ are in $H^{1,1}(T^4_I; \mathbb{R})$. We already know that B is also in $H^2(T^4; \mathbb{Q})$, when $(T^4; G, B)$ is for a rational (S)CFT.

If $\mathbb{R}(*B) = \mathbb{R}B$, then either $*B = B$ or $*B = -B$. In the case $B_{\parallel} \neq 0$, $*B = B$ is the only option, and $B_{\parallel} = B = *B$. In that situation, either B or $-B$ is a polarization. To see this, note first that $\int_{T^4} B \wedge B = \int_{T^4} B \wedge (*B) > 0$. This means that the Hermitian 2×2 matrix $(B_{a\bar{b}})$ in $B = iB_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$ has a positive determinant, so the product of the two eigenvalues of the matrix is positive. This proves that either B or $-B$ is positive definite, besides being rational.

If $*B$ and B are linearly independent in $H^2(T^4; \mathbb{R})$, then $\mathrm{Span}_{\mathbb{R}}\{B, *B\} \subset H^{1,1}(T^4; \mathbb{R})$ is a 2-dimensional subspace, with signature (1, 1). Now, we claim that $\mathbb{R}(*B) \cap H^2(T^4; \mathbb{Q})$ is not $\{0\}$. To see this, it is enough to note that

$$(*B)_{IJ} = \sqrt{\det(G)}\, \epsilon_{IJMN}\, G^{MK} G^{NL} B_{KL} \frac{1}{2}, \tag{3.4}$$

where ϵ_{IJKL} is the $\{\pm 1\}$-valued totally anti-symmetric tensor of rank-4; $\mathbb{R}(*B)$ contains such 2-forms as $\sqrt{\deg(G)}^{\pm 1}(*B)$, which are rational, as promised. This means that $\mathcal{H}^2(T^4_I) = H^2(T^4; \mathbb{Q}) \cap H^{1,1}(T^4_I; \mathbb{R})$ is at least of 2-dimensions over $\mathbb{Q}$ of signature (1, 1). Moreover, within the 2-dimensional $\mathcal{H}^2(T^4_I)$, the there is a line $\mathbb{R}\omega$ along the Kähler form, and there is a rational point of $\mathcal{H}^2(T^4_I)$ arbitrarily close to the line in $\mathcal{H}^2(T^4_I) \otimes \mathbb{R}$. Such a rational point is a polarization.

The last statement in Prop. 3.4 follows from Lemma 3.6 below. We review it below for the benefit of the reader not familiar with it. □

Such a complex structure in Prop. 3.4 is almost unique when $B_{\parallel} \neq 0$, and there will be infinitely many when $B_{\parallel} = 0$ (see the discussion at the end of section 3.1).

Lemma 3.6 (well known in math)**.** *Let $T_M \otimes \mathbb{Q}$ be the transcendental part of a Kähler surface M that has a polarization in $H^2(M; \mathbb{Q})$. When $\psi \in T_M \otimes \mathbb{Q}$ is decomposed into $\psi^{(2,0)} + \psi^{(0,2)} + \psi^{(1,1)}$ and $\psi^{(2,0)} = 0$, then $\psi = 0$.*

Proof: The input $\psi^{(2,0)} = 0$ implies $\psi^{(0,2)} = 0$, because $\psi \in T_M \otimes \mathbb{Q}$ is real. This means that $\psi = \psi^{(1,1)}$ is in $\mathcal{H}^2(M)$.

Since we have assumed that M admits a polarization, M is algebraic, so the intersection form on $\mathcal{H}^2(M)$ is non-degenerated (Hodge index theorem); $H^2(M; \mathbb{Q}) \cong \mathcal{H}^2(M) \oplus T_M \otimes \mathbb{Q}$ then. So, $\psi = \psi^{(1,1)}$ is both in $\mathcal{H}^2(M)$ and $T_M \otimes \mathbb{Q}$, which is possible only if $\psi = 0$. □

4 On the Rational Constant Kähler Metric

4.1 It is in the Algebraic Part

For a rational constant metric G on T^{2n} and a polarizable complex structure I with which G is compatible, there is an intriguing property on the Kähler form $\omega = 2^{-1}G(I-,-)$. In section 4.1 (and henceforth), any complex structure under consideration is always of this kind, even when the authors fail to mention that explicitly. To prove Thm. 4.5, we begin with this elementary preparation.

4.1.1 A Convenient Rational Basis

We have seen that the rational Hodge structure on $H^1(T_I^{2n};\mathbb{Q})$ is of CM type, when G is rational and compatible with a polarizable I. So, a $2n-$dimensional CM algebra $\oplus_{\alpha\in\mathcal{A}}F_\alpha$ over $\mathbb{Q}$ acts faithfully on the $2n$-dimensional vector space $H^1(T_I^{2n};\mathbb{Q})$. This can be used to introduce a rational basis of $H^1(T_I^{2n};\mathbb{Q})$ with which various computations are easier.

The idea is to use the fact explained in Lemma A.11; we can do so because the individual CM fields F_α act faithfully on their corresponding $[F_\alpha:\mathbb{Q}]$-dimensional vector subspaces of $[H^1(T^{2n};\mathbb{Q})]_\alpha$. In the case F is a CM field K with a primitive CM type, it often becomes convenient when we choose a basis $\{\eta'_{i=1,\cdots,[K:\mathbb{Q}]}\}$ of $K/\mathbb{Q}$ so that $\{\eta'_{i=1,\cdots,[K:\mathbb{Q}]/2}\}$ forms a basis of the totally real subfield K_0 of K, and use a purely imaginary generator ξ_* of the extension K/K_0 (i.e., $K = K_0(\xi_*)$ such that $(\xi_*)^2 \in K_0$) to fill the rest of a basis by $\{\eta'_{i+[K:\mathbb{Q}]/2} = (\xi_*\eta'_i) \mid i = 1,\cdots,[K:\mathbb{Q}]/2\}$. We apply this prescription to the cases (B, C) for $n = 2$; we use the basis $\{1, y, x, xy\}$ of $K/\mathbb{Q}$ as the basis $\{\eta'_i\}$ in Lemma A.11, and then there must be an appropriate rational basis $\{v'_i\}$ of $H^1(T^4;\mathbb{Q})$ such that $v'_i\tau_a(\eta'_i)$ for $a \in \{\pm\pm\}$ become the simultaneous eigenvectors of the action of the endomorphisms in $K \cong \mathrm{End}(H^1(T_I^4;\mathbb{Q}))^{\mathrm{Hdg}}$. The rational basis $\{v'_i\}$ corresponding to $\{\eta'_i\} = \{1, y, x, xy\}$ is denoted by $\{\hat{\alpha}^1, \hat{\alpha}^2, \hat{\beta}_1, \hat{\beta}_2\}$ in this article. It is further convenient to introduce two complex coordinates $z^{a=1,2}$ on T^4 so that $dz^1 = v_{a=++}$ and $dz^2 = v_{a=-+}$. Namely,

$$(dz^1, dz^2) = (\hat{\alpha}^1, \hat{\alpha}^2, \hat{\beta}_1, \hat{\beta}_2)\begin{pmatrix} 1 & 1 \\ \sqrt{d} & -\sqrt{d} \\ \sqrt{p+q\sqrt{d}} & \sqrt{p-q\sqrt{d}} \\ \sqrt{p+q\sqrt{d}}\sqrt{d} & -\sqrt{p-q\sqrt{d}}\sqrt{d} \end{pmatrix} =: (\hat{\alpha}^i, \hat{\beta}_i)\begin{pmatrix} Z^T \\ \alpha^T \end{pmatrix}; \quad (4.1)$$

Z is real-valued and α pure-imaginary valued; both are 2×2 matrices.

For the cases (A) and (A'), it is convenient to apply Lemma A.11 to the CM elliptic curves referred to in Lemma 2.22. In the case (A'), the imaginary quadratic field $K_i^{(2)} \cong \mathbb{Q}[x_i]/(x_i^2 - p_i)$ acts on the i-th elliptic curve of CM-type. We use the basis $\{\eta'\} = \{1, x_i\}$ of $K_i^{(2)}$, and Lemma A.11 ensure that there is a rational basis $\{v'\} = \{\hat{\alpha}^i, \hat{\beta}_i\}$ of $H^1(E_i; \mathbb{Q})$ so that $dz^i = \hat{\alpha}^i + \sqrt{p_i}\hat{\beta}_i$ is the (1,0) form, which is also an eigenvector of the action of the endomorphisms in $K_i^{(2)}$.

In the case (A), we may also choose a rational basis as $\{\hat{\alpha}^1, \hat{\beta}_1\} \cup \{\hat{\alpha}^2, \hat{\beta}_2\}$ in $H^1(E \times E; \mathbb{Q}) \cong H^1(E; \mathbb{Q}) \oplus H^1(E; \mathbb{Q})$, and introduce complex coordinates $z^{a=1,2}$ on the two CM elliptic curves E by $dz^1 = \hat{\alpha}^1 + \sqrt{p}\hat{\beta}_1$ and $dz^2 = \hat{\alpha}^2 + \sqrt{p}\hat{\beta}_2$.

In both cases (A') and (A), there must be an isogeny φ from the abelian variety $M = (T^4; I)$ to $E_1 \times E_2$ and $E \times E$, respectively. We pull back the convenient rational basis $\{\hat{\alpha}^i, \hat{\beta}_i\}$ of $E_1 \times E_2$ and $E \times E$ to $H^1(M; \mathbb{Q})$, respectively, and also pull back the complex coordinates $z^{1,2}$ to M, and use the same notation, $\{\hat{\alpha}^i, \hat{\beta}_i\}$ and $z^{1,2}$. In the cases (A') and (A),

$$(dz^1, dz^2) = (\hat{\alpha}^i, \hat{\beta}_i) \begin{pmatrix} Z^T \\ \alpha^T \end{pmatrix}, \qquad Z = \mathrm{diag}(1,1), \quad \alpha = \mathrm{diag}(\sqrt{p_1}, \sqrt{p_2}); \tag{4.2}$$

in the case (A), $\alpha = \mathrm{diag}(\sqrt{p}, \sqrt{p})$.

Note that the basis $\{\hat{\alpha}^1, \hat{\beta}_1, \hat{\alpha}^2, \hat{\beta}_2\}$ of $H^1(T^4; \mathbb{Q})$ chosen above is generically not a set of generators of the entire $H^1(T^4; \mathbb{Z}) \cong \mathbb{Z}^{\oplus 4}$. That is not a problem; the observation of GV [2] was that rational CFTs may be characterized by using a rational Hodge structure, not an integral Hodge structure, so we just need a rational basis. Although there are infinitely many mutually non-isomorphic CM-type abelian surfaces, they may have one of only three—(A), (A') and (B, C)—qualitatively different rational Hodge structures. Conveniently, all the analysis in this article needs to be performed for just these three cases.

4.1.2 The Algebraic and Transcendental Parts

We claim in Thm. 4.5 that the Kähler form ω is always in the algebraic part $\mathcal{H}^2(T_I^4) \otimes \mathbb{R}$. For this purpose, we need to know $\mathcal{H}^2(T_I^4)$.

Lemma 4.1. [well known in math literatures (e.g., [6, 37, 19])] *Let M be a complex torus of dimension n where $\mathrm{End}(H^1(M; \mathbb{Q}))^{\mathrm{Hdg}}$ is a CM field F. Then the algebraic part $\mathcal{H}^2(M) \subset H^2(M; \mathbb{Q})$ contains an n dimensional subspace $\mathcal{H}^2(M)_{\mathrm{gen}}$ specified below ($h^{1,1}(M) = n^2$, so that is possible).* The proof also introduces a basis on $\mathcal{H}^2(M)_{\mathrm{gen}}$ and also explains how to construct a polarization within $\mathcal{H}^2(M)_{\mathrm{gen}} \subset \mathcal{H}^2(M)$.

Proof: Let F_0 be the totally real subfield of F, and $\Phi = \{\tau_{a=1,\cdots,n}\}$ be the CM type corresponding to the Hodge (1, 0) components of $H^1(M;\mathbb{Q})$. There must be a basis $\{e_{I=1,\cdots,2n}\}$ of $H^1(M;\mathbb{Q})$ and a basis $\{\eta_{I=1,\cdots,2n}\}$ of $F/\mathbb{Q}$ so that $dz^a := e_I \tau_a(\eta_I)$ for $a = 1, \cdots, n$ are the n holomorphic 1-forms (cf Lemma A.11).

Now, let $\xi_* \in F$ be a generator of F/F_0 (i.e., $F = F_0(\xi_*)$) so that $\xi_*^2 \in F_0$. Then for any element $\xi \in \xi_* F_0^\times$,

$$\mathcal{Q}^{(\xi)} := \sum_{a=1}^{n} 2\tau_a(\xi) dz^a \wedge d\bar{z}^{\bar{a}}, \tag{4.3}$$

$$= e_I \wedge e_J \sum_{a=1}^{n} \left(\tau_a(\xi\eta_I\bar{\eta}_J) - \tau_a(\xi\bar{\eta}_I\eta_J)\right),$$

$$= e_I \wedge e_J \sum_{a=1}^{n} \left(\tau_a(\xi\eta_I\bar{\eta}_J) + \tau_a(\bar{\xi}\bar{\eta}_I\eta_J)\right) = e_I \wedge e_J \mathrm{Tr}_{F/\mathbb{Q}}[\xi\eta_I\bar{\eta}_J] \in \mathcal{H}^2(M) \tag{4.4}$$

(for any field extenstion E/F, $\mathrm{Tr}_{E/F}[x] \in F$ for $x \in E$; see [15, A.1.15] or any introductory textbook on field theory). Linearly independent choices of ξ from $\xi_* F_0$ generate an n-dimensional subspace of $\mathcal{H}^2(M)$, which is denoted by $\mathcal{H}^2(M)_{\mathrm{gen}}$.

For the (1, 1) form $\mathcal{Q}^{(\xi)}$ to be a polarization, first, choose $\{e_I\}$ to be an integral basis of $H^1(M;\mathbb{Z})$, and restrict to ξ such that $\mathrm{Tr}_{F/\mathbb{Q}}[\xi\eta_I\bar{\eta}_J] \in \mathbb{Z}$ for all the pairs (I, J); the basis $\{\eta_I\}$ should be those that correspond to the integral basis $\{e_I\}$. Second, impose inequalities on $\xi \in \xi_* F_0$ so that it is positive definite. $\square$

Lemma 4.2. *Let $M = (T^{2n}; I)$ be an abelian variety of CM-type. Then*

$$\dim_{\mathbb{Q}} \mathcal{H}^2(M) \geq \dim_{\mathbb{C}} M.$$

Proof: We can split the vector space $H^1(M;\mathbb{Q})$ into its components $\oplus_{a\in A}[H^1(M;\mathbb{Q})]_a$ supporting simple Hodge substructures; let K_a be the CM field $\mathrm{End}([H^1(M;\mathbb{Q})]_a)^{\mathrm{Hdg}}$. Thus, it is enough to prove the statement for a simple abelian variety, and that was done in Lemma 4.1. $\square$

Lemmas 4.1 and 4.2 above imply that

$$T_M \otimes \mathbb{C} \subset T_M^{\mathrm{gen}} \otimes \mathbb{C} = \mathrm{Span}_{\mathbb{C}} \left\{ (dz^a \wedge dz^b)_{a<b},\ (d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}})_{a<b},\ (dz^a \wedge d\bar{z}^{\bar{b}})_{a\neq b} \right\}, \tag{4.5}$$

$$\mathcal{H}^2(M) \otimes \mathbb{C} \supset \mathcal{H}^2(M)_{\mathrm{gen}} \otimes \mathbb{C} = \mathrm{Span}_{\mathbb{C}} \left\{ dz^a \wedge d\bar{z}^{\bar{a}} \right\} \tag{4.6}$$

for a CM abelian surface M. In fact,

4.3. This is for $n = 2$. **In the cases (B, C, A')** of CM abelian surfaces M, $\mathcal{H}^2(M)_{\rm gen}$ is the entire $\mathcal{H}^2(M)$; that is almost evident from the details below, but we justify this later in Discussion 5.2.

In the case (B, C),

$$dz^1 \wedge d\bar{z}^{\bar{1}} = -2\sqrt{p + q\sqrt{d}} \left\{ (\hat{\alpha}^1 \hat{\beta}_1) + d(\hat{\alpha}^2 \hat{\beta}_2) + \sqrt{d}(\hat{\alpha}^1 \hat{\beta}_2 + \hat{\alpha}^2 \hat{\beta}_1) \right\}, \tag{4.7}$$

$$dz^2 \wedge d\bar{z}^{\bar{2}} = -2\sqrt{p - q\sqrt{d}} \left\{ (\hat{\alpha}^1 \hat{\beta}_1) + d(\hat{\alpha}^2 \hat{\beta}_2) - \sqrt{d}(\hat{\alpha}^1 \hat{\beta}_2 + \hat{\alpha}^2 \hat{\beta}_1) \right\}, \tag{4.8}$$

so

$$\mathcal{H}^2(M) = \mathrm{Span}_{\mathbb{Q}} \left\{ (\hat{\alpha}^1 \hat{\beta}_1) + d(\hat{\alpha}^2 \hat{\beta}_2),\ (\hat{\alpha}^1 \hat{\beta}_2 + \hat{\alpha}^2 \hat{\beta}_1) \right\}, \tag{4.9}$$

$$T_M \otimes \mathbb{Q} = \mathrm{Span}_{\mathbb{Q}} \left\{ (\hat{\alpha}^1 \hat{\alpha}^2),\ (\hat{\beta}_1 \hat{\beta}_2),\ (\hat{\alpha}^1 \hat{\beta}_2 - \hat{\alpha}^2 \hat{\beta}_1),\ (\hat{\alpha}^1 \hat{\beta}_1 - d\hat{\alpha}^2 \hat{\beta}_2) \right\}. \tag{4.10}$$

A generator of the Hodge (2,0) component is in $T_M \otimes \mathbb{C}$, because

$$\begin{aligned} dz^1 \wedge dz^2 = -2\sqrt{d} \Big[& \hat{\alpha}^1 \hat{\alpha}^2 + (\sqrt{+} - \sqrt{-})/(2\sqrt{d})\ [\hat{\alpha}^1 \hat{\beta}_1 - d\hat{\alpha}^2 \hat{\beta}_2] \\ & + (\sqrt{+} + \sqrt{-})/2\ [\hat{\alpha}^1 \hat{\beta}_2 - \hat{\alpha}^2 \hat{\beta}_1] + \sqrt{d'}\ \hat{\beta}_1 \hat{\beta}_2 \Big]. \end{aligned} \tag{4.11}$$

Here, we used the notation introduced below (2.24).

In the case (A'),

$$dz^1 \wedge d\bar{z}^{\bar{1}} = -2\sqrt{p_1} \hat{\alpha}^1 \hat{\beta}_1, \qquad dz^2 \wedge d\bar{z}^{\bar{2}} = -2\sqrt{p_2} \hat{\alpha}^2 \hat{\beta}_2, \tag{4.12}$$

so $dz^1 \wedge dz^2 = (\hat{\alpha}^1 \hat{\alpha}^2) - \sqrt{p_1 p_2}(\hat{\beta}_1 \hat{\beta}_2) + \sqrt{p_2}(\hat{\alpha}^1 \hat{\beta}_2) + \sqrt{p_1}(\hat{\beta}_1 \hat{\alpha}^2)$ is in $T_M \otimes \mathbb{C}$ below:

$$\mathcal{H}^2(M) = \mathrm{Span}_{\mathbb{Q}} \left\{ (\hat{\alpha}^1 \hat{\beta}_1),\ (\hat{\alpha}^2 \hat{\beta}_2) \right\}, \tag{4.13}$$

$$T_M \otimes \mathbb{Q} = \mathrm{Span}_{\mathbb{Q}} \left\{ (\hat{\alpha}^1 \hat{\alpha}^2),\ (\hat{\beta}_1 \hat{\beta}_2),\ (\hat{\alpha}^1 \hat{\beta}_2),\ (\hat{\alpha}^2 \hat{\beta}_1) \right\}. \tag{4.14}$$

In the case (A),[25]

$$T_M \otimes \mathbb{Q} = \mathrm{Span}_{\mathbb{Q}} \left\{ (\hat{\alpha}^1 \hat{\alpha}^2 + p\hat{\beta}_1 \hat{\beta}_2),\ (\hat{\alpha}^1 \hat{\beta}_2 + \hat{\beta}_1 \hat{\alpha}^2) \right\}, \tag{4.15}$$

generated by the real and imaginary part of $dz^1 \wedge dz^2 = (\hat{\alpha}^1 + \sqrt{p}\hat{\beta}_1)(\hat{\alpha}^2 + \sqrt{p}\hat{\beta}_2)$.

$$\mathcal{H}^2(M) = \mathrm{Span}_{\mathbb{Q}} \left\{ (\hat{\alpha}^1 \hat{\beta}_1),\ (\hat{\alpha}^2 \hat{\beta}_2),\ (\hat{\alpha}^1 \hat{\alpha}^2 - p\hat{\beta}_1 \hat{\beta}_2),\ (\hat{\alpha}^1 \hat{\beta}_2 - \hat{\beta}_1 \hat{\alpha}^2) \right\}, \tag{4.16}$$

generated by $dz^1 \wedge d\bar{z}^{\bar{1}}/(-2\sqrt{p})$ and $dz^2 \wedge d\bar{z}^{\bar{2}}/(-2\sqrt{p})$ in $\mathcal{H}^2(M)_{\rm gen}$, along with the real and imaginary part of $dz^1 \wedge d\bar{z}^{\bar{2}}$. □

[25] The case (A), where M is isogenous to a product of two copies of a CM elliptic curve, is known [40] to be the case of rank-2 T_M.

4.4. An idea is explained in [19, Thm. 2.5] how to construct a rational constant Kähler metric on CM-type complex abelian variety M. It is done by decomposing M into its factors M_α where the algebra $\mathrm{End}(H^1(M_\alpha;\mathbb{Q}))^{\mathrm{Hdg}}$ contains a CM field F_α with $[F_\alpha : \mathbb{Q}] = 2\dim_{\mathbb{C}} M_\alpha$. Here, we review the construction, as we will refer to this construction (already in section 3.1 and also) later in section 4.1.3 and discussions in 5.11 and 5.13.

Let us use the same notation as in Lemma 4.1 here, except dropping the subscript α above. For any $\beta \in \xi_* F_0$,

$$\omega^{(\beta)} := \frac{i}{2}\sum_{a=1}^{n} \tau_a(\xi_*)\bar{\tau}_{\bar{a}}(\beta)\left(dz^a \otimes d\bar{z}^{\bar{a}} - d\bar{z}^{\bar{a}} \otimes dz^a\right), \tag{4.17}$$

$$G^{(\beta)} := \sum_{a=1}^{n} \tau_a(\xi_*\bar{\beta})\left(dz^a \otimes d\bar{z}^{\bar{a}} + d\bar{z}^{\bar{a}} \otimes dz^a\right), \tag{4.18}$$

$$= (e_I \otimes e_J)\sum_{a=1}^{n}\big(\tau_a(\xi_*\bar{\beta}\eta_I\bar{\eta}_J) + \tau_a(\xi_*\bar{\beta}\bar{\eta}_I\eta_J)\big) = e_I \otimes e_J\ \mathrm{Tr}_{F/\mathbb{Q}}[\xi_*\bar{\beta}\eta_I\bar{\eta}_J]. \tag{4.19}$$

In this construction, all the components $G^{(\beta)}_{IJ}$ are rational. One just has to impose inequalities on $\xi_*\bar{\beta} \in F_0 \hookrightarrow \mathbb{R}^n$ so that the metric is positive definite. In the application to the cases (B, C, A), we may set $\xi_* = x$ in Lemma 2.19; $F_0 = K_0$ in the case (B, C). In the application to the case (A'), we may set $\xi_* = x_i$ for $i = 1, 2$, and $F_0 = \mathbb{Q}$ for both $i = 1, 2$.

4.1.3 Analysis

The first one of the conditions in (2.4)—one for the metric—involves an integral basis of $H^1(T^{2n};\mathbb{Z})$. This condition is still in the same form—the components are rational numbers when we use a rational basis of $H^1(T^{2n};\mathbb{Q})$. As a preparation for the analysis in section 5, let us translate this condition by using the CM-algebra eigenstate basis $\{dz^a, d\bar{z}^{\bar{a}}\}$ of $H^1(T^{2n}_I;\mathbb{C})$.

The metric G is Hermitian under the complex structure I, that is,

$$G = h_{a\bar{b}} dz^a \otimes d\bar{z}^{\bar{b}} + h_{\bar{a}b} d\bar{z}^{\bar{a}} \otimes dz^b \tag{4.20}$$

for some constant Hermitian $n \times n$ matrix $h = (h_{a\bar{b}})$. Using the linear relations such as (4.1, 4.2), the rationality of the components G_{IJ} in a rational basis is translated to the rationality of all the components of the matrix

$$\begin{pmatrix} Z^T & \overline{Z}^T \\ \alpha^T & \overline{\alpha}^T \end{pmatrix}\begin{pmatrix} & h \\ h^T & \end{pmatrix}\begin{pmatrix} Z & \alpha \\ \overline{Z} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} Z^T h\overline{Z} + \overline{Z}^T h^T Z & Z^T h\overline{\alpha} + \overline{Z}^T h^T \alpha \\ \overline{\alpha}^T h^T Z + \alpha^T h\overline{Z} & \alpha^T h\overline{\alpha} + \overline{\alpha}^T h^T \alpha \end{pmatrix}. \tag{4.21}$$

That is,

$$Z^T h\overline{Z} + \overline{Z}^T h^T Z \ \in M_n(\mathbb{Q})^{\rm sym}, \tag{4.22}$$

$$\alpha^T h\overline{\alpha} + \overline{\alpha}^T h^T \alpha \ \in M_n(\mathbb{Q})^{\rm sym}, \tag{4.23}$$

$$Z^T h\overline{\alpha} + \overline{Z} h^T \alpha \ \in M_n(\mathbb{Q}). \tag{4.24}$$

We wish to solve those conditions in terms of the matrix $h = (h_{a\bar{b}})$; this is done by working separately for the cases (B, C), (A'), and (A) separately. So, the analysis leading to Thm. 4.5 is only for $n = 2$, $T^{2n=4}$. We will use the following parametrization of the 2×2 matrix h:

$$h = \left(\begin{array}{cc} h_1 & c_1 - ic_2' \\ c_1 + ic_2' & h_2 \end{array} \right), \qquad h_{1,2},\ c_1,\ c_2' \in \mathbb{R}. \tag{4.25}$$

Case (B, C): The condition (4.22) implies that

$$h_1 + h_2 \in \mathbb{Q}, \qquad h_1 - h_2 \in \sqrt{d}\mathbb{Q}, \qquad c_1 \in \mathbb{Q}, \qquad {}^{\forall}c_2' \in \mathbb{R}, \tag{4.26}$$

and the condition (4.23) on top of this implies that

$$c_1 = 0. \tag{4.27}$$

On the other hand, the condition (4.24) is equivalent to $c_2' = 0$ with arbitrary $h_{1,2}$ and c_1. So, we have[26]

$$h = \mathrm{diag}(a + b\sqrt{d}, a - b\sqrt{d}), \qquad {}^{\exists}a,\ b \in \mathbb{Q}; \tag{4.28}$$

the corresponding Kähler form $\omega(-,-) = 2^{-1}G(I-,-)$ is

$$\omega = i(a + b\sqrt{d})dz^1 \wedge d\bar{z}^{\bar{1}} + i(a - b\sqrt{d})dz^2 \wedge d\bar{z}^{\bar{2}}. \tag{4.29}$$

This family of Kähler forms parametrized by $a, b \in \mathbb{Q}$ is the same as the family (4.17) parametrized by $\beta \in \xi_* F_0$ ($[F_0 : \mathbb{Q}] = [K_0 : \mathbb{Q}] = 2$ in the case (B,C)).

Cases (A') and (A): The condition (4.22) is translated to $h_{1,2}, c_1 \in \mathbb{Q}$, and the condition (4.23) on top of this imposes $c_1 \in \sqrt{p_1 p_2}\mathbb{Q}$. So we should have $c_1 = 0$ in the case (A'), while $c_1 \in \sqrt{p_1 p_2}\mathbb{Q}$ is equivalent to $c_1 \in \mathbb{Q}$ in the case (A). On the other hand, the condition (4.24) implies $c_2' \in \sqrt{-p_1}\mathbb{Q} \cap \sqrt{-p_2}\mathbb{Q}$; so we should have $c_2' = 0$ in the case (A'), while we just have $c_2' \in \sqrt{-p}\mathbb{Q}$. To summarize, we should have

$$(A'): \qquad h = \mathrm{diag}(a_1, a_2), \qquad a_{1,2} \in \mathbb{Q}, \tag{4.30}$$

$$(A): \qquad h = \left(\begin{array}{cc} h_1 & c_1 - c_2\sqrt{p} \\ c_1 + c_2\sqrt{p} & h_2 \end{array} \right), \qquad h_{1,2},\ c_1,\ c_2 \in \mathbb{Q}, \tag{4.31}$$

[26]For the metric to be positive definite, $a > 0$ and $a^2 - b^2 d > 0$.

and the corresponding Kähler forms are

$$\omega \ = ia_1 dz^1 \wedge d\bar{z}^{\bar{1}} + ia_2 dz^2 \wedge d\bar{z}^{\bar{2}}, \tag{4.32}$$

$$\omega \ = i(dz^1, dz^2) \wedge \begin{pmatrix} h_1 & c_1 - c_2\sqrt{p} \\ c_1 + c_2\sqrt{p} & h_2 \end{pmatrix} \begin{pmatrix} d\bar{z}^{\bar{1}} \\ d\bar{z}^{\bar{2}} \end{pmatrix}, \tag{4.33}$$

respectively. The family of Kähler forms in (4.32) in the case (A') is the same as the family in (4.17); for the case (A), however, the full family of Kähler forms (4.33) corresponding to rational metrics has four rational parameters, whereas the family (4.17) constructed by using one CM subfield $F \subset \mathrm{End}(H^1(M;\mathbb{Q}))^{\mathrm{Hdg}}$ has $[F_0 : \mathbb{Q}] = 2$ rational parameters.

Having done this analysis, we are ready for this:

Theorem 4.5. This is for $n = 2$. *Let $(T^{2n=4}; G, B)$ be a set of data for which the (S)CFT is rational. For a polarizable complex structure I on T^4 with which G is compatible, the Kähler form $\omega = 2^{-1}G(I-,-)$ is always in the algebraic part of the 2-forms, $\mathcal{H}^2(T^4_I) \otimes \mathbb{R}$.*

Moreover, the combination $i\omega$ is in $\mathcal{H}^2(T^4_I) \otimes \tau^r_{(20)}(K^r)$, where K^r is the reflex field in 2.20 and 2.21, and $\tau^r_{(20)}$ its embedding for the Hodge (2, 0) component in $T_{T^4_I} \otimes \mathbb{C}$.

Proof: It is just necessary to write down the Kähler forms in (4.29, 4.32, 4.33) in the rational basis in Discussion 4.3. In the case (B, C),

$$i\omega \ = 2\tau^r_{++}(a\xi_r + bqd/\xi_r)e_1 + 2\tau^r_{++}(bd\xi_r + aqd/\xi_r)e_2, \tag{4.34}$$

$$e_1 := \hat{\alpha}^1\hat{\beta}_1 + d\hat{\alpha}^2\hat{\beta}_2, \quad e_2 := \hat{\alpha}^1\hat{\beta}_2 + \hat{\alpha}^2\hat{\beta}_1. \tag{4.35}$$

In the case (A'),

$$i\omega \ = 2a_1\sqrt{p_1}(\hat{\alpha}^1\hat{\beta}_1) + 2a_2\sqrt{p_2}(\hat{\alpha}^2\hat{\beta}_2), \tag{4.36}$$

while

$$i\omega \ = \sqrt{p}\left[2h_1(\hat{\alpha}^1\hat{\beta}_1) + 2h_2(\hat{\alpha}^2\hat{\beta}_2) + 2c_1(\hat{\alpha}^1\hat{\beta}_2 - \hat{\beta}_1\hat{\alpha}^2) + 2c_2(\hat{\alpha}^1\hat{\alpha}^2 - p\hat{\beta}_1\hat{\beta}_2)\right] \tag{4.37}$$

in the case (A). □

Prop. 4.1 and Cor. 5.11 of Ref. [19] identifies an example that looks like a counter example to the (spirit of the) conjecture in section 2.2. The example in [19, §4] chose a Kähler form within $H^{1,1}(M;\mathbb{R})$ but not in $\mathcal{H}^2(M) \otimes \mathbb{R}$, so there is no wonder in the light of Thm. 4.5 above that the metric is not rational in that example. This observation suggests that the Kähler form being in the algebraic part is an important element in characterizing the data for rational CFTs. So, this property is now implemented as the condition 2(b) of Thms. 5.8 and 5.9 to be meant as a revised version of Conjecture 2.1.

4.2 A Geometric Mirror Always Exists

Theorem 4.6. (This is only for the $n = 2$ cases) *Let* $(T^{2n=4}; G, B)$ *be a set of data for which the* $\mathcal{N} = (1,1)$ *SCFT is rational. Choose*[27] *a polarizable complex structure* I *with which* G *is compatible so that the Hodge (2,0) component of* B *is absent. Then one can always find a rank-*$n = 2$ *primitive subgroup* Γ_f *of* $H_1(T^4; \mathbb{Z})$ *so that the T-dual along* Γ_f *has a geometric SYZ-mirror.*

Proof: In the cases (B, C, A'), we have stated in 4.3 that $\hat{\alpha}^1\hat{\alpha}^2$ is in the transcendental part of $H^2(T^4; \mathbb{Q})$. So, when we choose $\Gamma_f \otimes \mathbb{Q} = \mathrm{Span}_{\mathbb{Q}}\{\alpha_1, \alpha_2\}$, $B|_{\Gamma_f} = 0$ because we chose the complex structure so that B is purely algebraic. We have also seen in Thm. 4.5 that the Kähler form is also in the algebraic part, so $\omega|_{\Gamma_f} = 0$.

In the case (A), we have seen in (4.31) how the Kähler form is parametrized. A rational $B = B^{\mathrm{alg}}$ should also have 4 parameters in $\mathbb{Q}$ because $\dim_{\mathbb{Q}} \mathcal{H}^2(T^4_I) = 4$. Details in 4.3 reveals that

$$B^{\mathrm{alg}} = \sqrt{p}(dz^1, dz^2) \wedge \begin{pmatrix} h_1^B & c_1^B - c_2^B\sqrt{p} \\ c_1^B + c_2^B\sqrt{p} & h_2^B \end{pmatrix} \begin{pmatrix} d\bar{z}^{\bar{1}} \\ d\bar{z}^{\bar{2}} \end{pmatrix}, \qquad h_{1,2}^B, c_{1,2}^B \in \mathbb{Q}. \tag{4.38}$$

With a straightforward computation, one can show that $\Gamma_f = \mathrm{Span}_{\mathbb{Z}}\{\alpha'_1, \alpha'_2\}$ with

$$\alpha'_1 = \alpha_1, \quad \alpha'_2 = \alpha_2 - \left\{ \frac{c_2}{h_1} - \frac{c_1}{h_1} \frac{(h_1^B c_2 - h_1 c_2^B)}{(h_1^B c_1 - h_1 c_1^B)} \right\} \beta^1 - \frac{h_1^B c_2 - h_1 c_2^B}{h_1^B c_1 - h_1 c_1^B} \beta^2 \tag{4.39}$$

satisfies the condition $\omega|_{\Gamma_f} = 0$ and $B|_{\Gamma_f} = 0$. In the case $h_1^B c_1 - h_1 c_1^B = 0$, the same conditions are satisfied when

$$\alpha'_1 = \alpha_1 + \frac{1}{pv}\beta^1, \qquad \alpha'_2 = \alpha_2 - \frac{c_1}{h_1}\left(v - \frac{1}{pv}\right)\beta^1 + v\beta^2, \qquad v \in \mathbb{Q},\ v \neq 0. \tag{4.40}$$

The T-dual in the directions (4.39, 4.40) have a geometric SYZ-mirror.

The choices of Γ_f above still come with a variety of choices of Γ_b, and moreover, there will be more choices of Γ_f other than the one above; they are meant to be only examples. $\square$

5 Refined Gukov–Vafa Theorem for T^4

5.1 CM Horizontal Hodge Structure

Let us quickly go through known results to confirm CM-type statements on horizontal rational Hodge structures. Whenever we refer to rational Hodge structures in this section 5.1, that is

[27] We have seen in section 3.2 that such a complex structure I exists. So, this Theorem is not empty for any set of data $(T^{2n}; G, B)$.

a horizontal one.

Prop. 3.3 already implies that the rational Hodge structure on $H^1(T^{2n}_I;\mathbb{Q})$ is of CM-type, when we choose a polarizable complex structure I on T^{2n} compatible with the rational metric G in a set of data $(T^{2n};G,B)$ for a rational SCFT. This may be combined with

Lemma 5.1. [25, Prop. 1.2] *Let (h_1,V_1) and (h_2,V_2) be both a polarizable rational Hodge structure. When both are of CM-type, then the polarizable rational Hodge structure $(h_2\otimes h_1, V_2\otimes V_1)$ on the vector space $V_2\otimes_{\mathbb{Q}} V_1$ is also CM type.* □

So, the polarizable rational Hodge structure on $\otimes^k(H^1(T^{2n}_I;\mathbb{Q})$ is also of CM-type, and its rational Hodge substructure on $\wedge^k H^1(T^{2n}_I;\mathbb{Q}) = H^k(T^{2n}_I;\mathbb{Q})$ is also of CM-type.[28]

Within $H^{k=n}(T^{2n}_I;\mathbb{Q})$, there is just one simple Hodge substructure of level-n, denoted by $[H^n(T^{2n}_I;\mathbb{Q})]_{\ell=n}$, which contains the 1-dimensional Hodge $(n,0)$ and $(0,n)$ components. The algebra $\mathrm{End}([H^n(T^{2n};\mathbb{Q})]_{\ell=n})^{\mathrm{Hdg}}$ should be a CM field, and moreover, the CM field should be[29] the reflex field K^r of the CM-type (F,Φ) of the CM-type rational Hodge structure $H^1(T^{2n}_I;\mathbb{Q})$.

5.2. In the case of $n=2$, let us confirm explicitly the general statements above; that also serves as a preparation for the discussion in section 5.3.

Let us begin with **the case (B, C)**. We may choose a generator of the Hodge $(n,0)=(2,0)$ component as $v_{++} := dz^1\wedge dz^2/(-2\sqrt{d})$; see (4.11) for the expression. This generator v_{++} is within the 4-dimensional space $T_M\otimes\mathbb{C}$ and is indeed in the form of $e_i\tau^r_{++}(\eta_i)$ for a rational basis $\{e_i\}$ given in (4.10) and a rational basis $\{\eta_i\}=\{1,q/\xi^r,\xi^r/2,y'\}$ of the reflex field K^r (see 2.20). It is also possible to confirm that $(dz^1\wedge d\bar{z}^{\bar{2}})$, $(d\bar{z}^{\bar{1}}\wedge dz^2)$ and $(d\bar{z}^{\bar{1}}\wedge d\bar{z}^{\bar{2}})$ are proportional to $e_i\tau^r_{-+}(\eta_i)$, $e_i\tau^r_{--}(\eta_i)$ and $e_i\tau^r_{+-}(\eta_i)$, respectively. The CM field K^r acts on the 4-dimensional vector space $T_M\otimes\mathbb{Q}$ in (4.10) as explained in Lemma A.12, which is in a way preserving the Hodge decomposition. So, $K^r\subset\mathrm{End}(T_M\otimes\mathbb{Q})^{\mathrm{Hdg}}$ indeed. The fact that K^r is not a CM algebra but a CM field implies that the 4-dimensional $T_M\otimes\mathbb{Q}$ as a whole (not its proper subspace) is the transcendental part indeed, and $\mathcal{H}^2(M)$ is no larger

[28] Any maximally commutative endomorphism (sub)algebra on $\wedge^k H^1(T^{2n};\mathbb{Q})$ and the one on the rest in $\otimes^k(H^1(T^{2n};\mathbb{Q}))$ can have dimensions as large as the dimension of the vector spaces they act. So, existence of a commutative subalgebra $F\subset\mathrm{End}(\otimes^k(H^1(T^{2n};\mathbb{Q})))^{\mathrm{Hdg}}$ with $\dim_{\mathbb{Q}}F=\dim_{\mathbb{Q}}\otimes^k(H^1(T^{2n};\mathbb{Q}))$ implies that there exists a commutative subalgebra $F_1\subset\mathrm{End}(\wedge^k(H^1(T^{2n};\mathbb{Q})))^{\mathrm{Hdg}}$ and F_2 on the rest of $\otimes^k(H^1(T^{2n};\mathbb{Q}))$ such that $\dim_{\mathbb{Q}}F_1=\dim_{\mathbb{Q}}\wedge^k(H^1(T^{2n};\mathbb{Q}))$ (to be rigorous, the structure (A.5) and the observation in footnote 21 need to be used).

[29] The authors could not identify a reference to cite for this statement, but an easy way to see this may be to note that the Hecke character associated with the holomorphic n-form is the product of the n Hecke characters associated with Φ. For string theorists, a direct computation as in 5.2 may be easier.

than $\mathcal{H}^2(M)_{\rm gen}$. We also note that the embedding τ^r_{++} of the number field K^r is the one associated with the Hodge (2,0) component (i.e., $\tau^r_{(20)}$ in Thm. 4.5).

In the **case A'**, literally the same argument as above can be repeated, by using 2.21 instead of 2.20, and using the rational basis (4.14) instead of (4.10). We may think of the embedding τ^r_{++} of the number field K^r as the one associated with the Hodge (2,0) component.

In the **case A**, $dz^1 \wedge dz^2 = (\hat{\alpha}^1\hat{\alpha}^2 - p\hat{\beta}_1\hat{\beta}_2) + \sqrt{p}(\hat{\alpha}^1\hat{\beta}_2 + \hat{\beta}_1\hat{\alpha}^2)$, which is expressed in the form of $e_i\tau^r(\eta_i)$ for the reflex field F^r in 2.20. The endomorphism algebra on the 2-dimensional $T_M \otimes \mathbb{Q}$ in 4.3 is a degree-2 extension field, so $T_M \otimes \mathbb{Q}$ is indeed the transcendental part.

Remark 5.3. Given the fact that the cohomology group of a Ricci-flat Kähler manifold M has a unique level-n simple Hodge substructure $[H^n(M;\mathbb{Q})]_{\ell=n}$, it is natural to wonder if this simple Hodge substructure plays more important role than other parts of the cohomology group. In the particular case of $M = T^{2n}$, for example, we may wonder whether or not a CM-type Hodge structure of $[H^2(T^{2n};\mathbb{Q})]_{\ell=n}$ implies that the rational Hodge structure on $H^1(T^{2n};\mathbb{Q})$ is CM-type.

In the case of $n = 2$, this question is split into four cases.

- (α) The transcendental part of $H^2(T^4_I;\mathbb{Q})$ is of 2-dimensions; its CM field K' should be $K' \cong \mathbb{Q}[\xi]/(\xi^2 - p)$ for $p \in \mathbb{Q}_{<0}$, an imaginary quadratic field.
- (α') The transcendental part is of 4-dimension, and its CM field K' is that of case (A); $K' \cong \mathbb{Q}[\xi, y']/(\xi^2 - p_1, (y')^2 - d) \cong \mathbb{Q}[\xi, (\xi y')]/(\xi^2 - p_1, (\xi y')^2 - dp_1)$; we may write $dp_1 =: p_2 \notin p_1(\mathbb{Q}^\times)^2$ because d is a square-free integer and $d \neq 1$.
- (β, γ) The transcendental part is of 4-dimension, and its CM field K' is that of cases (B, C).

The case (α) has been completely understood, and the question is answered affirmative [40] (as mentioned already in footnote 25). For other cases, certainly the CM abelian surfaces of case A', and B, C are examples of the cases α', and β, γ here. It is not obvious, however, whether or not all the abelian surfaces with the property α', β, γ are such CM abelian surfaces.

5.2 The Vertical Hodge Structure is of CM-type

5.2.1 Mirror Isogeny and Hodge Isomorphism

The following result by [19] exploits various properties very special to torus target SCFTs. For example, torus-target SCFTs forms a self-mirror moduli space of string vacua; all the data of constant metric G, closed 2-form B, complex structure I can be treated by constant-valued matrices than a field configuration, and all the information on cohomology and Hodge structure follows from that on $H^1(T^{2n};\mathbb{Q})$. Nevertheless, we use the result quoted as Prop. 5.4 and reach a nice-looking result (Thms. 5.8 and 5.9) that can be stated in a language that also (almost) makes sense for other families of target spaces. Now, let us begin with

Proposition 5.4. [19, Prop. 3.10] *Let $(T^{2n};G,B)$ be a set of data for which the $\mathcal{N}=(1,1)$ SCFT is rational, and I a complex structure with which G is compatible; here, I may or may not be polarizable. Suppose further that this SCFT with I has a geometric SYZ-mirror, with a set of data $(T^{2n};G^\circ,B^\circ;I^\circ)$. Then the complex tori T^{2n}_I and $T^{2n}_{I^\circ}$ are isogenous.*

Now, we can combine this Prop. 5.4 with Thm. 4.6 (verified for $n=2$, and for a polarizable I). For any $(T^{2n=4};G,B)$ that yields a rational $\mathcal{N}=(1,1)$ SCFT, there is a geometric SYZ-mirror, and there is an isomorphism of rational Hodge structures[30]

$$(W_h^k/W_h^{k+2}) \cong_{/\mathbb{Q}} g^*(W_{h^\circ}^k/W_{h^\circ}^{k+2}) \tag{5.1}$$

for all $k=0,1,\cdots,2n=4$; the Hodge structure is given by I (equivalently, by $\rho_{\rm spin}(h_{I,B})$) on the left-hand side, and by I° (equivalently, by $\rho_{\rm spin}(h_{\omega,B})$) on the right-hand side. By using Prop. 3.3 and the discussion right after Lemma 5.1, we arrive at the following theorem.

Theorem 5.5. This is[31] for $n=2$. *For a general $(T^{2n=4};G,B)$ whose corresponding $\mathcal{N}=(1,1)$ SCFT is rational, choose a polarizable complex structure I with which G is compatible, and $B^{(2,0)}=0$; such a complex structure I exists because of Prop. 3.4. For any geometric SYZ-mirror of $(T^{2n=4};G,B;I)$ (which is known to exist because of Thm. 4.6), with the T-dual taken along $\Gamma_f\subset H_1(T^{2n};\mathbb{Z})$, the vertical rational Hodge structure on $g^*(W_{h^\circ}^k/W_{h^\circ}^{k+2})$ by $\rho_{\rm spin}(h_{\omega,B})$ is of CM-type, and is Hodge isomorphic to the CM-type rational Hodge structure (W_h^k/W_h^{k+2}) by $\rho_{\rm spin}(h_{I,B})$ for all $k=0,1,\cdots,2n=4$.*

When there are multiple geometric SYZ-mirrors, there are multiple different filtrations $g^*(W_{h^\circ}^\bullet)$ installed on $H^*(T^{2n=4};\mathbb{Q})$. The statement here is meant to apply for one common $\rho_{\rm spin}(h_{\omega,B})$ and all different $g^*(W_{h^\circ}^\bullet)$. That is not surprising given the discussion in 2.16, however.

Thm. 5.5 is meant to be a refined version of [19, Thm. 3.11]. In the present version, we make it clear that this Thm. 5.5 is applicable and is not an empty statement for any set of data $(T^{2n};G,B;I)$ for a rational SCFT, albeit only for $n=2$ at this moment.

[30] This also means that the rational Hodge structures on $H^k(T_\circ^{2n=4};\mathbb{Q})$ by I° are polarizable, because we chose I so that the rational Hodge structure on $H^1(T^{2n};\mathbb{Q})$ is polarizable [28, Prop. 9.4.3].

[31] Since Thm. 4.6 has been confirmed only for $n=2$.

Remark 5.6. The CM-ness of the vertical rational Hodge structures on $g^*(W^k_{h\circ}/W^{k+2}_{h\circ})$ follows immediately from the rational nature of the mirror $\mathcal{N} = (1,1)$ SCFT,[32] combined with Props. 3.3 (and the discussions after Lemma 5.1). We need Prop. 5.4, however, for the existence of a horizontal–vertical Hodge isomorphism $(W^k_h/W^{k+2}_h) \cong g^*(W^k_{h\circ}/W^{k+2}_{h\circ})$.

5.2.2 The Simple Level-n Vertical Hodge Substructure

The states $e^{2^{-1}(B\pm i\omega)}$ in $H^*(T^{2n};\mathbb{Q})\otimes\mathbb{C}$ are the generators of the unique $h_{\omega,B}(e^{i\alpha}) = e^{\mp in\alpha}$ eigenstates (see Discussion 2.11). These U(1) eigenstates must be in $g^*(W^n_{h\circ}\otimes\mathbb{C})$ of any mirror description; we cannot claim that these states must be purely in $g^*(H^n(T^{2n}_\circ;\mathbb{C}))$, because it is not guaranteed whether the Hodge $(2,0)$ component (with respect to I°) of B° vanishes.

Choose one mirror description for definiteness for the moment. Then $\mho := e^{2^{-1}(B+i\omega)} \in g^*(W^2_{h\circ})$ has a decomposition

$$\mho = \mho_4 e^\circ_4 + \mho_2 \in g^*(H^4(T^4_\circ;\mathbb{C})) \oplus g^*(H^2(T^4_\circ;\mathbb{C})), \tag{5.2}$$

where $\mathbb{Q}e^\circ_4 = g^*(H^4(T^4_\circ;\mathbb{Q})) \subset H^*(T^4;\mathbb{Q})$. The decomposition is possible in fact within

$$\mho = \mho_4 e^\circ_4 + \mho_2 \in \tau^r_{(20)}(K^r)\otimes_\mathbb{Q} g^*(H^4(T^4_\circ;\mathbb{Q})) \oplus \tau^r_{(20)}(K^r)\otimes_\mathbb{Q} g^*(H^2(T^4_\circ;\mathbb{Q})), \tag{5.3}$$

because $B = B^{\mathrm{alg}}$ is rational, and $i\omega \in H^2(T^4;\mathbb{Q})\otimes\tau^r_{(20)}(K^r)$ as we have seen in Thm. 4.5. The vertical rational Hodge structure on $g^*(H^{n=2}(T^{2n=4}_{I^\circ};\mathbb{Q}))$ is of CM-type (Thm. 5.5), so there must be a CM-type simple Hodge substructure of level-$(n=2)$, $g^*([H^2(T^4_\circ;\mathbb{Q})]_{\ell=2}) \subset g^*(H^2(T^4_\circ;\mathbb{Q}))$; the state $\mho_2$ must be in this level-2 component. The CM field is the reflex field K^r because $\mathrm{End}(T_M\otimes\mathbb{Q})^{\mathrm{Hdg}} \cong K^r$. As a general property (Lemma A.11), the $\dim_\mathbb{Q}(T_M\otimes\mathbb{Q})$-dimensional level-$n$ simple Hodge substructure is generated by the Galois conjugates on the linear combination coefficients of the state[33] $\mho_2$ relatively to a rational basis of the $\dim_\mathbb{Q}(T_M\otimes\mathbb{Q})$-dimensional vector space; in fact, it does not matter any one of rational basis of the larger space $H^*(T^4;\mathbb{Q})$ is used for the expansion. The $[K^r:\mathbb{Q}]$ states $\{\mho_2^\sigma \mid \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$ generate the vector space $g^*([H^2(T^4_\circ;\mathbb{Q})]_{\ell=2})\otimes\mathbb{C}$. See Fig. 2. The argument here

[32] The $\mathcal{N}=(1,1)$ SCFT for $(T^{2n};G,B)$ and $\mathcal{N}=(1,1)$ SCFT for $(T^{2n};G^\circ,B^\circ)$ are isomorphic. So the latter is rational when the former is.

[33] In applying Lemma A.11, we should keep in mind that we should rescale the state $\mho_2$ to $\mho_2' \in \mathbb{C}\mho_2$ in general so that $\mho_2'$ is identified with a state of the form $e_I\tau^r_a(\eta_I)$ for some rational basis $\{e_I\}$ of the $[K^r:\mathbb{Q}]$-dimensional vector space $g^*([H^2(T^4_\circ;\mathbb{Q})]_{\ell=2})$ and some basis $\{\eta_I\}$ of $K^r/\mathbb{Q}$. In the application here, however, we already know that $\mho_2 \in \tau^r_{(20)}(K^r)\otimes H^*(T^4;\mathbb{Q})$, so we should use it as it is for the state of the form $e_I\tau^r(\eta_I)$ in Lemma A.11.

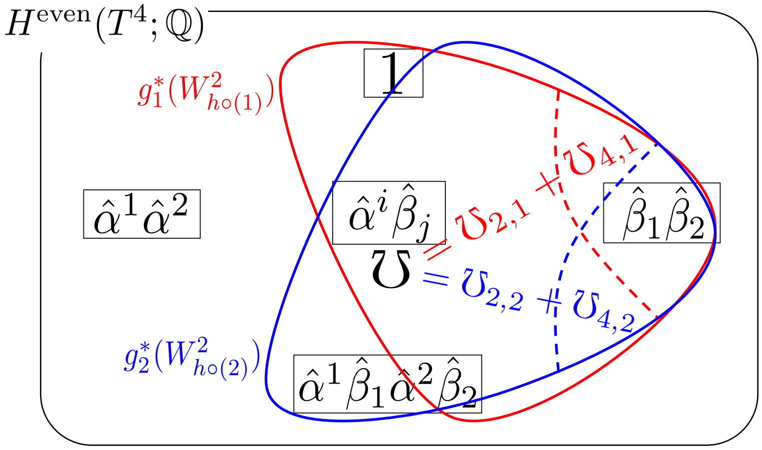

Figure 2: The grading structures pulled back from multiple different SYZ-mirrors are not necessarily identical. The decomposition of $\mho \in H^*(T^4;\mathbb{C})$ into $\mho_4 e_4^\circ$ and $\mho_2$ therefore depends on the SYZ-mirrors. The combination $\mho$ is still within $g^*(W^2_{h\circ}\otimes\mathbb{C})$ for any geometric SYZ-mirror.

may sound a little abstract; it is still a straightforward exercise to work out the decomposition $\mho = \mho_4 e_4^\circ + \mho_2$ for each example of geometric SYZ-mirrors in section 4.2.

It follows from the discussion above that the $[K^r : \mathbb{Q}]$ states are in this subspace,

$$\mho^\sigma = \mho_4^\sigma e_4^\circ + \mho_2^\sigma \in g^*(H^4(T^4_\circ;\mathbb{C})) \oplus g^*([H^2(T^4_\circ;\mathbb{C})]_{\ell=2}) \subset \mathbb{C}\otimes g^*(W^2_{h\circ}), \tag{5.4}$$

while they also generate a $[K^r : \mathbb{Q}]$-dimensional subspace—denoted by $g^*(T_M^\circ)\otimes\mathbb{C}$—of the algebraic part $A(T^4_I)\otimes\mathbb{C} \subset H^*(T^4;\mathbb{C})$, where

$$A(M)\otimes\mathbb{Q} := \oplus_{m=0}^n \left(H^{2m}(M;\mathbb{Q}) \cap H^{m,m}(M;\mathbb{R})\right) \tag{5.5}$$

for a Kähler manifold M. The composition

$$g^*(T_M^\circ\otimes\mathbb{Q}) \hookrightarrow g^*(W^2_{h\circ}) \to g^*([H^2(T^4_\circ;\mathbb{Q})]_{\ell=2}), \qquad \mho^\sigma \longmapsto \mho_2^\sigma \tag{5.6}$$

is an isomorphism of rational Hodge structures. This isomorphism is in fact that of a polarized Hodge structure, because the pairing (2.22) is not sensitive to the $W^4_{h\circ}$ component. Therefore, reminding ourselves that the discussion above holds true for any mirror description, we have

Theorem 5.7. *Let $(T^{2n=4};G,B)$ be a set of data for a rational $\mathcal{N}=(1,1)$ SCFT, and I a polarizable complex structure with which G is compatible and $B^{(2,0)}=0$. Then there exists a*

$[K^r:\mathbb{Q}]$-dimensional vector subspace $g^*(T_M^\circ)\otimes\mathbb{Q}\subset A(T_I^4)\otimes\mathbb{Q}$ determined uniquely, which admits a CM-type weight-2 polarized rational Hodge structure with the endomorphism field K^r; its Hodge $(2,0)$ and $(0,2)$ components are generated by $e^{2^{-1}(B+i\omega)}$, and the polarization is given by (2.22).

For any geometric SYZ-mirror, $g^*(T_M^\circ)\otimes\mathbb{Q}\subset g^*(W_{h\circ}^{n=2})$, and there is an isomorphism of polarized rational Hodge structures between $g^*(T_M^\circ)\otimes\mathbb{Q}$ and $g^*([H^2(T_\circ^4;\mathbb{Q})]_{\ell=2})$. $\square$

For a general set of data $(T^{2n};G,B)$ for which the $\mathcal{N}=(1,1)$ SCFT is not necessarily rational, and for a general complex structure I with which G is compatible, $\dim_{\mathbb{Q}}(A(T_I^{2n}))$ can be as small as 2. On the other hand, generically the level-n simple Hodge substructure of $H^n(T_{I^\circ}^{2n};\mathbb{Q})$, if a geometric SYZ-mirror exists, is of $b_n={}_{2n}C_n$ dimensions. Obviously the property stated in Thm. 5.7 no longer holds true in such a general set up.

Let us now summarize what we have found so far for T^4 in the style as close as possible to Conjecture 2.1.

Theorem 5.8. Let $(M;G,B)$ a set of data of a real $2n$-dimensional manifold M, a Ricci-flat metric G and a closed 2-form B on M; we assume that there exists a complex structure I so that (M,G,I) can be regarded as a Kähler manifold. We have so far verified the following statements in the case $M=\mathbb{R}^4/\mathbb{Z}^{\oplus 4}=T^4$.

Suppose that the $\mathcal{N}=(1,1)$ SCFT for the set of data $(M;G,B)$ is rational. Then

1. there exists a polarizable complex structure I on M, with which G is compatible and (M,G,I) becomes Kähler, and $B^{\rm transc}=0$.

For such a complex structure I (M_I is meant to be the complex manifold (M,I)), there are

2. properties on the horizontal and vertical simple level-n rational Hodge substructures:
   (a) The level-n simple Hodge substructure on $[H^n(M;\mathbb{Q})]_{\ell=n}$ by I is of CM-type, where the CM field $\mathrm{End}([H^n(M;\mathbb{Q})]_{\ell=n})^{\rm Hdg}$ is denoted by K'.
   (b) There exists a $[K':\mathbb{Q}]$-dimensional vector subspace $A(M_I)\otimes\mathbb{Q}$ denoted by $T_M^v\otimes\mathbb{Q}$ on which a simple level-n rational Hodge structure of weight-n can be introduced, with the polarization (2.22); its Hodge $(n,0)$ component is generated by $\mho := e^{2^{-1}(B+i\omega)}$, where $\omega=2^{-1}G(I-,-)$, and this polarized rational Hodge structure is of CM-type, with the endomorphism field K'.
   (c) There is an isomorphism of polarized rational Hodge structure of weight-n between the vertical and horizontal simple level-n components $T_M^v\otimes\mathbb{Q}$ and $T_M\otimes\mathbb{Q}$.

3. There are also properties on the rational Hodge substructures other than the level-n components:
   (a) All other rational polarizable Hodge structures on $H^k(M;\mathbb{Q})$ by I are also of CM-type.
   (b) There is a filtration $W_v^\bullet$ on $H^*(M;\mathbb{Q})$ so that the data $(\rho_{\rm spin}(h_{\omega,B}), W_v^\bullet)$ introduces a generalized rational Hodge structure on $H^*(M;\mathbb{Q})$, so that
      i. $T_M^v \otimes \mathbb{Q} \subset W_v^n$, and
      ii. the rational Hodge structures on W_v^k/W_v^{k+2} is of CM-type for all k, and the one for $k=n$ is polarized by the pairing (2.22).

Furthermore,

4. there is a geometric SYZ-mirror to the $\mathcal{N}=(2,2)$ SCFT for the data $(M;G,B;I)$, and
5. the filtration $W_v^\bullet$ referred to above (and no.6 below) can be interpreted as that of $g^*(W_{h\circ}^\bullet)$, the one on the geometric SYZ-mirror.

Finally, there is one more property that makes sense only for a family of $(M;G,B)$ that is self-SYZ-mirror (as in the case of $M=T^{2n}$ and K3):

6. there is a one-to-one correspondence between the simple rational horizontal Hodge substructures on (W_h^k/W_h^{k+2}) and vertical Hodge substructures on (W_v^k/W_v^{k+2}) so that there are Hodge isomorphism.

Furthermore, here is one more property whose generalization to M other than a torus is not obvious:

7. the isomorphisms between the horizontal and vertical rational Hodge structures can be interpreted as a combination of an isogeny and a mirror map of D-brane charges.

At this moment, the authors do not have hard evidence to believe that the 3rd property follows as a consequence of the 2nd one, because an issue remains in Rmk. 5.3. So, the properties no.2 and no.3 are listed independently here.

Proof: This is just a brief note on the origin of the properties above. The property no.1 is from Prop. 3.4. The property no.2(a) is essentially due to [19, Thm. 2.5] (quoted as Prop. 3.3 in this article), while the fact that $B+i\omega \in \mathcal{H}^2(T^4)\otimes \tau^r_{(n0)}(K^r)$—2.(b)—is from Prop.

3.4 and Thm. 4.5. The combination of the properties 3.(b).i and 5, $T_M^v \otimes \mathbb{Q} \subset g^*(W_{h\circ}^n)$ for any geometric SYZ-mirror, is from the discussion leading to Thm. 5.7.

For a target space M other than tori and K3 surfaces, the charge n state $\mho$ under $\rho_{\rm spin}(h_{\omega,B})$ is no longer $e^{2^{-1}(B+i\omega)}$, but is likely to be the one including the worldsheet instanton corrections. That is indicated at least by the example of the Gepner construction of a Fermat quintic Calabi–Yau threefold (reviewed in section 2.2). □

5.2.3 Trial Statements for a Complex Structure with $B^{(2,0)} \neq 0$

The results in Thms. 5.5, 5.7 and 5.8 are valid only for a polarizable complex structure I where $B^{(2,0)} = 0$, because Thm. 4.6 guaranteeing a geometric SYZ-mirror has been proven only for such complex structures. For a general polarizable complex structure I, not necessarily $B^{(2,0)} = 0$, one may still be interested in characterizing the data $(T^{2n=4}; G, B; I)$ for a rational SCFT in terms of Hodge structures. Thm. 5.9 below is meant to be for that broader class of complex structures.

Suppose that $(T^4; G, B)$ is for a rational $\mathcal{N} = (1, 1)$ SCFT, and I a polarizable complex structure with which G is compatible. Suppose further that a geometric SYZ-mirror exists. Then Prop. 3.3, Thms. 4.5 and 5.5 still hold true. Discussion leading to Thm. 5.7, however, needs to be modified at one point. The elements $\{\mho^\sigma\}$ do not generate a $[K^r : \mathbb{Q}]$-dimensional subspace of $A(M)\otimes\mathbb{C}$, but a $[K^r : \mathbb{Q}]$-dimensional subspace of $(e^{B^{\rm transc}/2}A(M)\otimes\mathbb{C})$; let us still use the same notation $g^*(T_M^\circ) \otimes \mathbb{Q}$ for the $[K^r : \mathbb{Q}]$-dimensional space in $e^{B^{\rm transc}/2}A(M) \otimes \mathbb{Q}$. So, we have an analogue of Thm. 5.8:

Theorem 5.9. Let $(M; G, B)$ a set of data of a real $2n$-dimensional manifold M, a Ricci-flat metric G and a closed 2-form B on M; we assume that there exists a complex structure I so that (M, G, I) can be regarded as a Kähler manifold. We have so far verified the following statements in the case $M = \mathbb{R}^4/\mathbb{Z}^{\oplus 4} \cong T^4$. In the following, we avoid repeating literally the same sentences as in Thm. 5.8 and abbreviate them by ".......".

Suppose that the $\mathcal{N} = (1, 1)$ SCFT for the set of data $(M; G, B)$ is rational. Choose a polarizable complex structure I on M with which G is compatible and (M, G, I) becomes Kähler, and assume that there is a geometric SYZ-mirror for $(M; G, B; I)$. Then

1. $B^{\rm transc} \in T_M \otimes \mathbb{Q} \subset T_M \otimes \mathbb{R}$.

For the complex structure I (M_I is meant to be the complex manifold (M, I)), thre are

2. properties on the horizontal and vertical simple level-n rational Hodge substructures:

(a) The level-n simple Hodge substructure on $[H^n(M;\mathbb{Q})]_{\ell=n}$ by I is of CM-type, where ....

(b) There exists a $[K':\mathbb{Q}]$-dimensional vector subspace of $e^{B^{\rm transc}/2}A(M_I)\otimes\mathbb{Q}$ denoted by $T^v_M\otimes\mathbb{Q}$ on which a simple level-n rational .... [34]

(c) There is an isomorphism of ....

3. There are also properties on the rational Hodge substructures other than the level-n components:

   (a) All other rational polarizable Hodge ....

   (b) There is a filtration $W^\bullet_v$ on $H^*(M;\mathbb{Q})$ so that ....

Furthermore,

4. (ignore this, because we did not prove Thm. 4.6 for I with $B^{(2,0)}\neq 0$)

5. the filtration $W^\bullet_v$ referred to above (and also at no.6 below) can be interpreted as ....

Finally, there is one more property that makes sense only for a family of $(M;G,B)$ that is self-SYZ-mirror (as in the case of $M=T^{2n}$ and K3):

6. there is a one-to-one correspondence between ....

Furthermore, here is one more property whose generalization to M other than ....:

7. the isomorphisms between the horizontal and vertical ...

The remark at the end of Thm. 5.8 also applies here. $\square$

Remark 5.10. Although the rationality of $B^{\rm transc}$ is listed (as the no.1 property) separately in the statement of Thm. 5.9, there may be a way to encode this property together with others.[35]

[34] The state $\mho=e^{2^{-1}(B+i\omega)}$ is in $e^{2^{-1}B^{\rm transc}}A(M_I)\otimes\mathbb{C}$, and the state $e^{2^{-1}(B^{\rm alg}+i\omega)}$ is in $A(M_I)\otimes\mathbb{C}$. The latter state is a U(1) eigenstate of $S^1_{\omega,B'}$ for $B'=B^{\rm alg}$. We are not sure if there is any importance in this observation, but let us just note this down here.

[35] The property 2(b), saying that there exists a vector subspace $(T^v_M\otimes\mathbb{Q})\subset H^*(M;\mathbb{Q})$ even within $e^{B^{\rm transc}/2}A(M_I)\otimes\mathbb{Q}$, makes sense only when $B^{\rm transc}$ is rational, though.

A naïve idea (which fails in the following) is to introduce the algebra of endomorphisms of a generalized rational Hodge structure,

$$\mathrm{End}(H^*(T^{2n};\mathbb{Q}))^{(\rho_{\mathrm{spin}}(h_{I,B}),W_h^\bullet)}, \tag{5.7}$$

those that preserve the filtration, and commute with the $\rho_{\mathrm{spin}}(h1_{I,B})$ action. When B^{transc} is rational, the algebra above contains

$$e^{B^{\mathrm{transc}}/2}\left(\oplus_k \mathrm{End}(H^k(T^{2n};\mathbb{Q}))^{\mathrm{Hdg}}\right)e^{-B^{\mathrm{transc}}/2}. \tag{5.8}$$

So, when the rational Hodge structures on $H^k(M;\mathbb{Q})$ by I are all CM-type, and B^{transc} is rational, the algebra (5.7) contains a commutative semi-simple algebra F over $\mathbb{Q}$ of dimension equal to $\dim_{\mathbb{Q}}(H^*(T^{2n};\mathbb{Q}))$ whose quotient representation on W_h^k/W_h^{k+2} is a $\mathbb{Q}$-algebra of dimension $b_k(T^{2n})$.

Conversely, however, it is possible for the case where B^{transc} differs from a rational $B'_{\mathrm{ratnl}} \in T_M \otimes \mathbb{Q}$ by a (1, 1) form, that the algebra (5.7) contains a commutative subalgebra F with $\dim_{\mathbb{Q}} F = \dim_{\mathbb{Q}}(H^*(T^{2n};\mathbb{Q}))$, and $\dim_{\mathbb{Q}}(F|_{W_h^k/W_h^{k+2}}) = b_k$, if $(T^{2n};I)$ is a complex torus with sufficiently many complex multiplications.

5.3 The Converse

Let us now study whether the converse of Thm. 5.8 is true. By imposing the 2nd and 3rd properties[36] in Thm. 5.8 on a set of data $(T^{2n=4};G,B)$ and a polarizable I with $B^{\mathrm{transc}}=0$ (implicitly, the 1st property is imposed as well), we will see that there is a significant likelihood that the resulting $\mathcal{N}=(1,1)$ SCFT is rational. However, there are data $(T^4;G,B;I)$ satisfying the two properties, and yet the corresponding CFTs are not rational. The following analysis is carried out separately for the cases (B, C), (A'), and (A).

5.3.1 Case (B, C)

Because of the 2nd and 3rd properties in the statement of Thm. 5.8, we have an abelian variety $M=(T^4,I)$ of CM-type; a CM field K of case (B, C) acts on $H^1(M;\mathbb{Q})$, also on $H^3(M;\mathbb{Q})$, and its reflex field K^r acts on the $[K^r:\mathbb{Q}]=4$-dimensional transcendental part $T_M\otimes\mathbb{Q}\subset H^2(M;\mathbb{Q})$. There is also $[K^r:\mathbb{Q}]$-dimensional vector subspace $T_M^v\otimes\mathbb{Q}\subset A(M)\otimes\mathbb{Q}$ on which $\rho_{\mathrm{spin}}(h_{\omega,B})$ introduces a rational Hodge structure with CM by K^r, polarized under the pairing (2.22). The combination $(B+i\omega)$ is in $[(T_M^v\otimes\mathbb{Q})\cap H^2(T^4;\mathbb{Q})]\otimes\mathbb{C}$, and $e^{2^{-1}(B\pm i\omega)}$

[36]Since it is not guaranteed whether the third property follows from the second, we impose both.

with the sign + and − should generate the Hodge (2,0) and (0,2) components, respectively. Let us exploit all those information and see whether one can claim that the B-field and the metric G are rational (the answer is no).

5.11. The fact that $\mho := e^{(B+i\omega)/2}$ is the only generator of the Hodge $(2,0)$ component of the CM-type Hodge structure on $T^v_M \otimes \mathbb{Q}$, with the CM field K^r and embedding $\tau^r_{(20)} = \tau^r_{++}$, implies that there must be a basis $\{1, \eta_1, \eta_2, \eta_4\}$ of $K^r/\mathbb{Q}$, so that

$$\mho \; = \tau^r_{++} \left[1 + e_1\eta_1 + e_2\eta_2 + (\hat{\alpha}^1\hat{\beta}_1\hat{\alpha}^2\hat{\beta}_2)\eta_4 \right], \tag{5.9}$$

where a rational basis $\{e_1, e_2\}$ of $\mathcal{H}^2(M)$ is the one introduced in (4.35); we must set $\eta_4 = (d\eta_1^2 - \eta_2^2) \in K^r$ so that $(\mho, \mho) = 0$. So, there are eight rational parameters for $\eta_1, \eta_2 \in K^r$, for the moment. The Hodge (0,2) component should be given by

$$\overline{\mho} = \tau^r_{+-} \left[1 + e_1\eta_1 + e_2\eta_2 + (\hat{\alpha}^1\hat{\beta}_1\hat{\alpha}^2\hat{\beta}_2)\eta_4 \right], \tag{5.10}$$

and the (1,1) components by the two vectors

$$\Sigma \; = \tau^r_{-+} \left[1 + e_1\eta_1 + e_2\eta_2 + (\hat{\alpha}^1\hat{\beta}_1\hat{\alpha}^2\hat{\beta}_2)\eta_4 \right], \tag{5.11}$$

$$\overline{\Sigma} \; = \tau^r_{--} \left[1 + e_1\eta_1 + e_2\eta_2 + (\hat{\alpha}^1\hat{\beta}_1\hat{\alpha}^2\hat{\beta}_2)\eta_4 \right]. \tag{5.12}$$

This Hodge decomposition must be polarized with respect to (2.22). The condition $(\mho, \mho) = 0$ is built in by construction, $\mho = e^{2^{-1}(B+i\omega)}$. The remaining non-trivial information from the polarization is that $(\mho, \Sigma) = 0$ and $(\mho, \overline{\Sigma}) = 0$. The two conditions are equivalent to

$$-2^{-1} \left(\tau^r_{++}(X) - \tau^r_{-\pm}(X), \tau^r_{++}(X) - \tau^r_{-\pm}(X) \right)_{\mathcal{H}^2} = 0 \tag{5.13}$$

for $X = e_1\eta_1 + e_2\eta_2$, using just the pairing in $\mathcal{H}^2(M)$; those conditions are further rewritten as

$$d \left(\tau^r_{++}(\eta_1) - \tau^r_{-\pm}(\eta_1) \right)^2 - \left(\tau^r_{++}(\eta_2) - \tau^r_{-\pm}(\eta_2) \right)^2 = 0 \tag{5.14}$$

in the normal closure of the number field K^r.

The eight rational parameters for $\eta_{1,2} \in K^r$, that is, $A, B, C, D, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in \mathbb{Q}$ in

$$\eta_1 =: A + By' + C\xi^r + D\xi^r y', \qquad \eta_2 =: \widetilde{A} + \widetilde{B}y' + \widetilde{C}\xi^r + \widetilde{D}\xi^r y', \tag{5.15}$$

should satisfy the conditions (5.14). Straightforward computation translates the conditions to

$$dBC = \widetilde{B}\widetilde{C}, \quad dBD = \widetilde{B}\widetilde{D}, \quad d(D^2d' - C^2) = (\widetilde{D}^2d' - \widetilde{C}^2), \tag{5.16}$$

along with

$$d\left[d'(B^2 - 2CD) + p(C^2 + d'D^2)\right] = \left[d'(\widetilde{B}^2 - 2\widetilde{C}\widetilde{D}) + p(\widetilde{C}^2 + d'\widetilde{D}^2)\right]. \tag{5.17}$$

There are four conditions on the eight parameters.

First, one can immediately see that the rational parameters A and $\widetilde{A}$ dropped out. So any $A, \widetilde{A} \in \mathbb{Q}$ has no conflict with the condition (5.14) for the consistency of the Hodge structure (of $B + i\omega$) with the polarization (2.22).

Second, we prove that $\widetilde{B} = 0$ by contradiction. If $\widetilde{B} \neq 0$, then $\widetilde{C}$ and $\widetilde{D}$ can be solved in terms of C, D and $B/\widetilde{B}$. Then

$$(D^2d' - C^2)\left(\frac{B^2}{\widetilde{B}^2}d - 1\right) = 0. \tag{5.18}$$

This is a contradiction[37] because neither d nor d' is a square of a rational number.

Thirdly, $\widetilde{B} = 0$ implies that either $B = 0$ or $C = D = 0$ holds true. The latter is not possible, however, because $\widetilde{D}^2d' - \widetilde{C}^2 = 0$ would follow, although d' is not a square of a rational number. So, $B = 0$. We have now proved that the B-field is

$$\tau^r_{++}\left[e_1(A + By') + e_2(\widetilde{A} + \widetilde{B}y')\right] = Ae_1 + \widetilde{A}e_2 \tag{5.19}$$

for free $A, \widetilde{A} \in \mathbb{Q}$. This is the same as saying that the B-field is in $\mathcal{H}^2(M)$. The rationality condition of the B-field (2.4) follows from the 2nd and 3rd (and 1st) properties in the statements of Thm. 5.8.

Next, change the parametrization as follows.

$$C = \frac{1}{2}\left(C' + \frac{pD'}{qd}\right), \quad D = \frac{D'}{2qd}, \quad \widetilde{C} = \frac{1}{2}\left(\widetilde{C}' + \frac{p}{q}\widetilde{D}'\right), \quad \widetilde{D} = \frac{\widetilde{D}'}{2q}, \tag{5.20}$$

or equivalently,

$$\xi^r(C + Dy') = D'\frac{q}{\xi^r} + \frac{C'}{2}\xi^r, \qquad \xi^r(\widetilde{C} + \widetilde{D}y') = \frac{\widetilde{C}'}{2}\xi^r + \widetilde{D}'\frac{qd}{\xi^r}. \tag{5.21}$$

[37] If $D = C = \widetilde{D} = \widetilde{C} = 0$, then $[T^v_M \otimes \mathbb{C}]^{(2,0)} = [T^v_M \otimes \mathbb{C}]^{(0,2)} = \mathbb{C} \subset T^v_M \otimes \mathbb{C}$. This is not appropriate as a Hodge decomposition. In physics terminology, this corresponds to $\omega = 0$, and volume$(T^4) = 0$.

Then the remaining two conditions on $C, D, \widetilde{C}, \widetilde{D}$ are rewritten as

$$d(C')^2 + \frac{2p}{q}(C'D') + (D')^2 \ = d(\widetilde{D}')^2 + \frac{2p}{q}\widetilde{C}'\widetilde{D}' + (\widetilde{C}')^2, \tag{5.22}$$

$$(C')^2 pd + (D'C')2qd + (D')^2 p \ = (\widetilde{D}')^2 dp + (\widetilde{D}'\widetilde{C}')2qd + (\widetilde{C}')^2 p. \tag{5.23}$$

So, this is equivalent to

$$D'C' = \widetilde{C}'\widetilde{D}', \qquad d(C')^2 + (D')^2 = d(\widetilde{D}')^2 + (\widetilde{C}')^2. \tag{5.24}$$

This coupled quadratic equations seem to allow two possibilities,

$$\frac{\widetilde{C}'}{\widetilde{D}'} = d\frac{C'}{D'}, \qquad \frac{\widetilde{C}'}{\widetilde{D}'} = \frac{D'}{C'}, \tag{5.25}$$

including $D' = \widetilde{D}' = 0$ and $\widetilde{D}' = C' = 0$, respectively. The first case is impossible, because $(\widetilde{C}')^2 = d(C')^2$ is a contradiction for the parameters $C', \widetilde{C}' \in \mathbb{Q}$ for d that is not a square. The only option is

$$(C', D') = (\widetilde{D}', \widetilde{C}'), \tag{5.26}$$

and

$$(C', D') = -(\widetilde{D}', \widetilde{C}'). \tag{5.27}$$

There are two kinds of solutions, (5.26) and (5.27), for the Hodge structure on $[T^v_M \otimes \mathbb{Q}]$ to be compatible with the polarization (2.22); for solutions of both kinds, there are two free rational parameters $C', D' \in \mathbb{Q}$ for ω (besides the two free parameters $A, \widetilde{A} \in \mathbb{Q}$ for the B-field).

In the first kind of solutions, (5.26), we have

$$\frac{i}{2}\omega \ = \tau^r_{++}\left(\frac{C'}{2}\xi^r + \frac{D'q}{\xi^r}\right)e_1 + \tau^r_{++}\left(\frac{C'qd}{\xi^r} + \frac{D'}{2}\xi^r\right)e_2, \tag{5.28}$$

$$= \frac{C'}{2}\left(e_1(\sqrt{+} + \sqrt{-}) + e_2(\sqrt{+} - \sqrt{-})\sqrt{d}\right) \tag{5.29}$$

$$+ \frac{D'}{2}\left(e_2(\sqrt{+} + \sqrt{-}) + e_1(\sqrt{+} - \sqrt{-})/\sqrt{d}\right),$$

$$= i(p\omega^{(x)} - q\omega^{(xy)})\left(-\frac{C'}{4d'}\right) + i(-qd\omega^{(x)} + p\omega^{(xy)})\left(-\frac{D'}{4dd'}\right). \tag{5.30}$$

The last expression is a rational linear combination of the basis $\omega^{(\beta)}$ of Kähler forms (4.17) corresponding to a rational metric; the expression in the middle can easily be identified with $i\omega$ in (4.34) for a rational metric with the dictionary $2a = C'$ and $2b = D'/d$.

In the other kind of solutions, (5.27), on the other hand,

$$\frac{i}{2}\omega = \tau^r_{++}\left(\frac{C'}{2}\xi^r + \frac{D'q}{\xi^r}\right)e_1 - \tau^r_{++}\left(\frac{C'qd}{\xi^r} + \frac{D'}{2}\xi^r\right)e_2, \tag{5.31}$$

$$= \frac{C'}{2}\left(e_1(\sqrt{+} + \sqrt{-}) - e_2(\sqrt{+} - \sqrt{-})\sqrt{d}\right) \tag{5.32}$$

$$+ \frac{D'}{2}\left(-e_2(\sqrt{+} + \sqrt{-}) + e_1(\sqrt{+} - \sqrt{-})/\sqrt{d}\right),$$

Rewriting this in terms of $dz^1 \wedge d\bar{z}^{\bar{1}}$ and $dz^2 \wedge d\bar{z}^{\bar{2}}$ according to (4.7) and (4.8),

$$\frac{i}{2}\omega = \frac{p - q\sqrt{d}}{4\sqrt{d'}}\left(C' - \frac{D'}{\sqrt{d}}\right)dz^1 \wedge d\bar{z}^{\bar{1}} + \frac{p + q\sqrt{d}}{4\sqrt{d'}}\left(C' + \frac{D'}{\sqrt{d}}\right)dz^2 \wedge d\bar{z}^{\bar{2}}. \tag{5.33}$$

For this Kähler form to be fitted by the expression (4.34), we have

$$a = -\frac{1}{2\sqrt{d'}}(pC' + qD'), \qquad b = \frac{1}{2d\sqrt{d'}}(qdC' + pD'). \tag{5.34}$$

The fitted parameters a, b are not rational when $C', D' \in \mathbb{Q}$. The metric corresponding to this Kähler form does not satisfy the condition (2.4). The resulting metric is positive definite for some region in $(C', D') \in \mathbb{Q}^2$, so the second kind of solutions include physically sensible $\mathcal{N} = (1, 1)$ SCFTs that are not rational. $\square$

5.12. We have restricted $(B + i\omega)$ by demanding that the vertical Hodge structure on $T^v_M \otimes \mathbb{Q}$ is of CM-type, with CM by the reflex field K^r of a CM field K. For such a $(B + i\omega)$, whether a solution of (5.26) or (5.27), the vertical Hodge structure on $H^1(T^4; \mathbb{Q}) \oplus H^3(T^4; \mathbb{Q})$ is already determined. Demanding that this rational Hodge structure is also of CM-type and that their endomorphism field is K, we find in the following that no extra condition is found on the parameters $A, \widetilde{A}, C', D' \in \mathbb{Q}$.

The charge $p - q = +1$ components in $H^1(T^4; \mathbb{Q}) \oplus H^3(T^4; \mathbb{Q})$ can be generated by $e^{2^{-1}(B+i\omega)}\hat{\alpha}^i$ and $e^{2^{-1}(B+i\omega)}\hat{\beta}_i$ with $i = 1, 2$; the charge $p - q = -1$ components are generated by the ones with $(B + i\omega)$ replaced by $(B - i\omega)$, as stated in 2.11. Now, we use

$$\frac{1}{2}(B + i\omega) = \tau^r_{++}\left(A + \frac{C'}{2}\xi^r + \frac{D'}{2}\frac{2q}{\xi^r}\right)e_1 + \tau^r_{++}\left(\tilde{A} \pm \frac{D'}{2}\xi^r \pm \frac{C'}{2}\frac{2qd}{\xi^r}\right)e_2, \tag{5.35}$$

where the $+$ and $-$ choices of $\pm$ correspond to the solution (5.26) and (5.27), respectively.

Within the vector space $V_1 := \mathrm{Span}_{\mathbb{Q}}\{\hat{\alpha}^1,\ \hat{\alpha}^2\} \oplus \mathrm{Span}_{\mathbb{Q}}\{\hat{\alpha}^1\hat{\alpha}^2\hat{\beta}_1,\ \hat{\alpha}^1\hat{\alpha}^2\hat{\beta}_2\}$,

$$(e^{2^{-1}(B+i\omega)}\hat{\alpha}^1, e^{2^{-1}(B+i\omega)}\hat{\alpha}^2) = \left(\hat{\alpha}^1,\ \hat{\alpha}^2,\ \hat{\alpha}^1\hat{\alpha}^2\hat{\beta}_1,\ \hat{\alpha}^1\hat{\alpha}^2\hat{\beta}_2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ Z_2 & -Z_1 \\ dZ_1 & -Z_2 \end{pmatrix}, \tag{5.36}$$

where

$$Z_1 := \tau^r_{++}\left(A + \frac{C'}{2}\xi^r + \frac{D'}{2}\frac{2q}{\xi^r}\right), \qquad Z_2 := \tau^r_{++}\left(\tilde{A} \pm \frac{D'}{2}\xi^r \pm \frac{C'}{2}\frac{2qd}{\xi^r}\right). \tag{5.37}$$

One finds the following structure when the two generators in $[V_1 \otimes \mathbb{C}]^{p-q=+1}$ are rearranged as follows:

$$\begin{aligned}(e^{2^{-1}(B+i\omega)}\hat{\alpha}^1,\ e^{2^{-1}(B+i\omega)}\hat{\alpha}^2) &\begin{pmatrix} 1 & 1 \\ \mp\sqrt{d} & \pm\sqrt{d} \end{pmatrix} \qquad (5.38) \\ &= \left(\hat{\alpha}^1,\ \hat{\alpha}^2,\ \hat{\alpha}^1\hat{\alpha}^2\hat{\beta}_1,\ \hat{\alpha}^1\hat{\alpha}^2\hat{\beta}_2\right)\begin{pmatrix} 1 & 1 \\ \tau_{++}(\mp y) & \tau_{-+}(\mp y) \\ \tau_{++}(\Xi_\pm) & \tau_{-+}(\Xi_\pm) \\ \tau_{++}(\pm\Xi_\pm y) & \tau_{-+}(\pm\Xi_\pm y) \end{pmatrix},\end{aligned}$$

where

$$\Xi_\pm = \widetilde{A} \pm Ay \pm D'x \pm C'xy \in K. \tag{5.39}$$

Unless $C' = D' = 0$ (which we are not interested because the volume of T^4 is precisely zero), $\{1, y, \Xi_\pm, \Xi_\pm y\}$ forms a basis of $K/\mathbb{Q}$. So, with Lemma A.12, we see that an algebra isomorphic to K acts on the vector space V_1 while preserving this Hodge decomposition.

A similar calculation can be carried out for the rest of the vector space, $V_3 := \mathrm{Span}_{\mathbb{Q}}\{\hat{\beta}_1, \hat{\beta}_2\} \oplus \mathrm{Span}_{\mathbb{Q}}\{\hat{\alpha}^1\hat{\beta}_1\hat{\beta}_2,\ \hat{\alpha}^2\hat{\beta}_1\hat{\beta}_2\}$ in $H^1(T^4;\mathbb{Q}) \oplus H^3(T^4;\mathbb{Q})$. The charge $p - q = +1$ components are generated by

$$\begin{aligned}\left(\hat{\beta}_1 e^{2^{-1}(B+i\omega)},\ \hat{\beta}_2 e^{2^{-1}(B+i\omega)}\right) &\begin{pmatrix} 1 & 1 \\ \mp\sqrt{d} & \pm\sqrt{d} \end{pmatrix} \qquad (5.40) \\ &= \left(\hat{\beta}_1,\ \hat{\beta}_2,\ \hat{\alpha}^1\hat{\beta}_1\hat{\beta}_2,\ \hat{\alpha}^2\hat{\beta}^1\hat{\beta}_2\right)\begin{pmatrix} 1 & 1 \\ \tau_{++}(\mp y) & \tau_{-+}(\mp y) \\ \tau_{++}(-\Xi_\pm) & \tau_{-+}(-\Xi_\pm) \\ \tau_{++}(\mp\Xi_\pm y) & \tau_{-+}(\mp\Xi_\pm y) \end{pmatrix}.\end{aligned}$$

The endomorphism algebra of V_3 contains K, so this Hodge structure is also of CM-type (the property 3.(b)ii).

The 2nd and 3rd properties in the statement of Thm. 5.8 allows one to choose/find a filtration $W_v^\bullet$ (the 5th property demands more, though). So, we choose $W_v^3 := V_3$. Then the horizontal rational Hodge structure on $H^3(T^4;\mathbb{Q})$ (resp. on $H^1(T^4;\mathbb{Q})$) is isomorphic to the vertical rational Hodge structure on W_v^3 (resp. W_v^1/W_v^3) (the property 6), as claimed at the beginning of 5.12. □

5.3.2 Case (A')

Let us work on the case the endomorphism algebra of $H^1(M;\mathbb{Q})$ is the one in case (A'). Let us exploit just the 1st, 2nd and 3rd properties of a pair of horizontal and vertical Hodge structures in the statement of Thm. 5.8 and see whether one can claim that B and G are rational (the answer is no). The logic and the procedure of the analysis are precisely the same as for the case (B, C). So, we will focus on small difference in the following presentation, and often avoid repeating the same logic.

5.13. Let us impose on $(B+i\omega)$ the conditions that the vertical Hodge structure on $T_M^v \otimes \mathbb{Q}$ is of CM type, with the endomorphism field K^r in 2.21. The generator $\mho := e^{2^{-1}(B+i\omega)}$ of the Hodge (2, 0) component of $T_M^\circ \otimes \mathbb{C}$ must be in the form of

$$\mho = \tau^r_{++}\left[1 + (\hat{\alpha}^1\hat{\beta}_1)\eta_1 + (\hat{\alpha}^2\hat{\beta}_2)\eta_2 + (\hat{\alpha}^1\hat{\beta}_1\hat{\alpha}^2\hat{\beta}_2)\eta_4\right] \tag{5.41}$$

for some basis $\{1, \eta_1, \eta_2, \eta_4\}$ of $K^r/\mathbb{Q}$. The property $(\mho, \mho) = 0$ implies that $\eta_4 = \eta_1\eta_2$, so there are eight rational parameters for η_1 and η_2 at this moment.

Let us parametrize the freedom by $A, B, C, D, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in \mathbb{Q}$, where

$$\eta_1 = A + By' + C\xi^r + D\xi^r y', \qquad \eta_2 = \widetilde{A} + \widetilde{B}y' + \widetilde{C}\xi^r + \widetilde{D}\xi^r y'. \tag{5.42}$$

For the Hodge decomposition to be compatible with its polarization (2.22), we impose $(\mho, \Sigma) = 0$ and $(\mho, \overline{\Sigma}) = 0$. As a result, we obtain

$$B\widetilde{B} + p_1 D\widetilde{D} = 0, \quad B\widetilde{D} + \widetilde{B}D = 0, \tag{5.43}$$

$$B\widetilde{B}p_2 + C\widetilde{C} = 0, \quad B\widetilde{C} + \widetilde{B}C = 0. \tag{5.44}$$

Now, we have four conditions on the eight rational parameters.

One can prove that $B = \widetilde{B} = 0$ (or otherwise we should accept an unphysical zero-volume situation (such as $C = D = 0$)); the proof is similar to the case (B, C), so we omit the detail.

The four conditions above are reduced to $D\widetilde{D} = 0$ and $C\widetilde{C} = 0$. So, there are two kinds of solutions (apart from the zero-volume situations):

$$
\begin{aligned}
D = 0, \quad \widetilde{C} = 0, \text{ so } 2^{-1}(B + i\omega) \qquad &= \hat{\alpha}^1\hat{\beta}_1(A + C\sqrt{p_1}) + \hat{\alpha}^2\hat{\beta}_2(\widetilde{A} + \widetilde{D}\sqrt{p_1}\sqrt{d'}), && (5.45)\\
C = 0, \quad \widetilde{D} = 0, \text{ so } 2^{-1}(B + i\omega) \qquad &= \hat{\alpha}^1\hat{\beta}_1(A + D\sqrt{p_1}\sqrt{d'}) + \hat{\alpha}^2\hat{\beta}_2(\widetilde{A} + \widetilde{C}\sqrt{p_1}). && (5.46)
\end{aligned}
$$

Therefore, the B field has to be rational ($B = \widetilde{B} = 0$ and $^\forall A, \widetilde{A} \in \mathbb{Q}$) for both kinds of the solutions (5.45, 5.46).

The first kind of solutions (5.45) reproduces all the rational metric G in (4.30, 4.36); $a_1 \sim C$ and $a_2 \sim \widetilde{D}(-p_1)$. In the second kind of solutions (5.46), the metric is not rational; $a_1 \sim D\sqrt{d'} \notin \mathbb{Q}$, and $a_2 \sim \widetilde{C}\sqrt{p_1/p_2} \notin \mathbb{Q}$. There is a region with a positive volume interpretation in the (a_1, a_2) space. $\square$

5.14. The combination $(B + i\omega)$ is parametrized by four rational parameters. For such a $(B+i\omega)$, the vertical Hodge structure is also given to $H^1(T^4;\mathbb{Q}) \oplus H^3(T^4;\mathbb{Q})$. For both kinds of solutions, (5.45) and (5.46), the vertical Hodge structure is also of CM-type, and there exists Hodge isomorphism with the horizontal Hodge structure on $H^1(T^4;\mathbb{Q}) \oplus H^3(T^4;\mathbb{Q})$, as we see below.

The vector space $V_1 := \mathrm{Span}_{\mathbb{Q}}\{\hat{\alpha}^1, \hat{\alpha}^2\} \oplus \mathrm{Span}_{\mathbb{Q}}\{\hat{\alpha}^1\hat{\alpha}^2\hat{\beta}_1,\ \hat{\alpha}^1\hat{\alpha}^2\hat{\beta}_2\}$ can be split into $V_{11} := \mathrm{Span}_{\mathbb{Q}}\{\hat{\alpha}^1,\ \hat{\alpha}^1\hat{\alpha}^2\hat{\beta}_2\}$ and $V_{12} := \mathrm{Span}_{\mathbb{Q}}\{\hat{\alpha}^2,\ \hat{\alpha}^2\hat{\alpha}^1\hat{\beta}_1\}$, and

$$
\begin{aligned}
&(\text{solution } (5.45)) : \qquad \mathrm{End}_{\mathbb{Q}}(V_{11})^{\mathrm{Hdg}} \cong K_2^{(2)}, \qquad \mathrm{End}_{\mathbb{Q}}(V_{12})^{\mathrm{Hdg}} \cong K_1^{(2)}, && (5.47)\\
&(\text{solution } (5.46)) : \qquad \mathrm{End}_{\mathbb{Q}}(V_{11})^{\mathrm{Hdg}} \cong K_1^{(2)}, \qquad \mathrm{End}_{\mathbb{Q}}(V_{12})^{\mathrm{Hdg}} \cong K_2^{(2)}. && (5.48)
\end{aligned}
$$

So, as a whole,

$$
\mathrm{End}_{\mathbb{Q}}(V_1)^{\mathrm{v.Hdg}} \cong K_1^{(2)} \oplus K_2^{(2)} \cong \mathrm{End}_{\mathbb{Q}}(H^1(T_I^4;\mathbb{Q}))^{\mathrm{Hdg}} \tag{5.49}
$$

for both solutions. The vertical rational Hodge structure on V_1 is of CM-type, with the CM field $K_1^{(2)} \oplus K_2^{(2)}$.

Similarly, $V_3 := \mathrm{Span}_{\mathbb{Q}}\{\hat{\beta}_1, \hat{\beta}_2\} \oplus \mathrm{Span}_{\mathbb{Q}}\{\hat{\alpha}^2\hat{\beta}_2\hat{\beta}_1,\ \hat{\alpha}^1\hat{\beta}_1\hat{\beta}_2\}$ can also be split into $V_{31} := \mathrm{Span}_{\mathbb{Q}}\{\hat{\beta}_1,\ \hat{\beta}_1\hat{\alpha}^2\hat{\beta}_2\}$ and $V_{32} := \mathrm{Span}_{\mathbb{Q}}\{\hat{\beta}_2,\ \hat{\beta}_2\hat{\alpha}^1\hat{\beta}_1\}$, and

$$
\begin{aligned}
&(\text{solution } (5.45)) : \qquad \mathrm{End}_{\mathbb{Q}}(V_{31})^{\mathrm{Hdg}} \cong K_2^{(2)}, \qquad \mathrm{End}_{\mathbb{Q}}(V_{32})^{\mathrm{Hdg}} \cong K_1^{(2)}, && (5.50)\\
&(\text{solution } (5.46)) : \qquad \mathrm{End}_{\mathbb{Q}}(V_{31})^{\mathrm{Hdg}} \cong K_1^{(2)}, \qquad \mathrm{End}_{\mathbb{Q}}(V_{32})^{\mathrm{Hdg}} \cong K_2^{(2)} && (5.51)
\end{aligned}
$$

for both of the solutions. So, the vertical rational Hodge structure on V_3 is also of CM-type, with the CM field $K_1^{(2)} \oplus K_2^{(2)}$.

The 2nd and 3rd properties of Thm. 5.8 does not restrict how one introduces a filtration $W_v^\bullet$, so we may still choose $W_v^3 = V_{31} \oplus V_{32}$. Then there is a Hodge isomorphism between the horizontal $H^3(T^4;\mathbb{Q})$ (resp. $H^1(T^4;\mathbb{Q})$) and the vertical W_v^3 (resp. W_v^1/W_v^3) (the 6th property in Thm. 5.8) for both kinds of the solutions. $\square$

5.3.3 Case (A)

Finally, let us work on the case (A). In this case, we find that $(B+i\omega)$ that satisfies the 1st, 2nd and 3rd properties in the statement of Thm. 5.8 correspond to rational B and G, and hence for a rational $\mathcal{N}=(1,1)$ SCFT.

5.15. In this case, the reflex field K^r is a degree-2 extension field $\mathbb{Q}(\sqrt{p})$. So, the 2-dimensional subspace $T_M^v \otimes \mathbb{Q} \subset A(M)\otimes\mathbb{Q}$ should be such that both $e^{2^{-1}(B+i\omega)}$ and $e^{2^{-1}(B-i\omega)}$ are contained in $T_M^v \otimes \mathbb{C}$. For the vertical Hodge structure on this space to be of CM by the degree-2 field $\mathbb{Q}(\sqrt{p})$, the generator $\mho$ of the charge $p-q=2$ component should be of the form

$$\mho = e^{2^{-1}(B+i\omega)} = \left(1 + B/2 + E(\hat{\alpha}^1\hat{\beta}_1\hat{\alpha}^2\hat{\beta}_2)\right)\tau_+^r(1) + \left(\omega'/2 + E'(\hat{\alpha}^1\hat{\beta}_1\hat{\alpha}^2\hat{\beta}_2)\right)\tau_+^r(\xi^r) \quad (5.52)$$

for some $E, E' \in \mathbb{Q}$ and B, ω' in the image of $T_M^v \otimes \mathbb{Q} \subset A(M)\otimes\mathbb{Q}$ projected into $\mathcal{H}^2(T_I^4)$. The rational constants E, E' are determined by B and ω' by the condition $(\mho,\mho) = 0$ in $T_M^v \otimes \mathbb{Q} \subset A(M)\otimes\mathbb{Q}$ with respect to the pairing (2.22). Free choice of $B \in \mathcal{H}^2(M)$ corresponds to a rational B-field in (2.4), and a free choice of $\sqrt{p}\omega' \in \sqrt{p}\mathcal{H}^2(M)$ corresponds to $i\omega$ for ω given in (4.37), which is for a rational metric. So, the first two properties in Thm. 5.8 are strong enough in the case (A) to allow only the data (G,B) for a rational CFT. $\square$

6 Discussions

In this article, we have made an attempt at refining Gukov–Vafa's conjecture, Conj. 2.1, and verifying it for the simple case where the target space is T^4. Rational CFTs in this case have been completely classified, so we used that to refine the conditions to be imposed (criteria) for rational SCFTs in the language of the horizontal and vertical rational Hodge structures.

As a result, we arrived at Thms. 5.8 and 5.9 stated in the language that is applicable, for their most part, to SCFTs with a general Ricci-flat Kähler manifold as the target space. Thm. 5.8 extracts properties that *all* the T^4-target RCFTs satisfy. The property 2(b) in

Thm. 5.8 rules out an example of a non-rational T^4-target CFT in [19, §4] that looks as if it were a counter example to the version Conjecture 2.1.

We have also found by imposing the properties 1, 2, 3, and 6 in Thm. 5.8 on a data $(T^4; G, B; I)$, that there are still T^4-target $\mathcal{N} = (1, 1)$ SCFTs that are not rational. Obviously it is one of the next steps to see whether those counter examples can be eliminated by implementing the 5th and 7th properties of Thm. 5.8.

Although the statements of Thm. 5.8 are phrased (as much as possible) in a way applicable to Ricci-flat Kähler manifolds, we do not make a clear stance on which subset of the itemized properties in Thm. 5.8 should be imposed as criteria for rationalness of the SCFTs. Some of the properties might be derived from others (cf Rmk. 5.3). Some of the properties may not hold true in some examples of rational SCFTs (such as 3.(a) and 3.(b)ii; cf discussions 2.2, 2.3 and footnote 6). Since there is still room for experimental study as in this article, the authors do not feel obliged to decide now which subset of the properties are necessary conditions for the rationalness.

There is a chance of having the 6th property as a part of necessary criteria for rational SCFTs, only for a family of Ricci-flat Kähler target SCFTs that are self-SYZ-mirror. We need to make an effort in properly formulating and generalizing the 7th property of Thm. 5.8 to Ricci-flat Kähler manifolds other than tori.

One can also enjoy a moment of speculation. Consider K3-target $\mathcal{N} = (1, 1)$ SCFTs. The most naïve way to apply the refined version Thm. 5.8 for this class of target space is to take the 1st, 2nd, 4th and 5th properties in Thm. 5.8 as the necessary and sufficient conditions for the SCFT to be rational. The 3rd and 6th properties do not contain additional information in this case. An immediate consequence of this is that there exists a complex structure on K3 such that there is a polarization, and at the same time $B^{(2,0)} = 0$ when a K3-target SCFT is rational. This speculation/conjecture is already non-trivial. Moreover, the Picard number ρ should be no less than 10, and the Kähler form must be within the image of $T^v_M \otimes \mathbb{R} \subset A(M) \otimes \mathbb{R}$ projected on to the Neron–Severi space $\mathcal{H}^2 \otimes \mathbb{R}$ for any K3 target rational SCFT.[38]

There has been a question of how densely rational SCFTs populate the moduli space of Ricci-flat Kähler target SCFTs. Reference [2] conjectured that rational SCFTs might have

[38] Reference [21] discusses how to find an appropriate B-field and a symplectic form for a complex CM-type K3 surface X with $\rho(X) \geq 10$ such that X has a mirror that is also of CM-type, motivated by the GV conjecture [2].

something to do with CM-type rational Hodge structures (Conj. 2.1), and further combined the observation with André–Oort conjecture in math [41, 42, 43],[39] which says that CM points are not very dense in the moduli space of such manifolds in general (except for the moduli space of abelian varieties and K3 surfaces). So, it has been hinted that rational SCFTs do not populate densely within the whole moduli space of $\mathcal{N} = (1, 1)$ SCFTs with a Calabi–Yau threefold target space. The experimental study carried out in this article concluded that all the rational (T^4-target) SCFTs satisfy the properties of Thm. 5.8 refined from Conj. 2.1. So, the inference on scarcity of rational SCFTs *from* the scarcity of CM-type Calabi–Yau manifolds does not have to be questioned at this moment. Finer understanding on the remaining issues listed above (e.g., footnote 6), however, might also change this perspective in the future.

The question above may have a consequence beyond mathematical physics. Suppose one day that mankind discovers that Type IIB flux compactification is theoretically consistent only when the SCFT is rational; it is not bad to enjoy such a speculation sometimes [2]. That may indicate that the vacuum complex structure of the internal Calabi–Yau threefold is something captured by a special subvariety of Calabi–Yau moduli space interpreted as a Shimura variety, if we speculate along the lines of Gukov–Vafa and Andrè–Oort. When the moduli space has a group action, discrete and/or continuous, its isotropy subgroup at the vacuum point may remain in the low-energy effective field theory of the moduli fields as gauged and/or accidental symmetry.

Acknowledgments

We thank H. Lange for useful communications. We thank M. Ashwinkumar and M. Yamazaki for collaboration during early stages of this research project. AK thanks DESY (Hamburg) for hospitality during the final stage of this project. This work was supported in part by WPI Initiative (all the authors), the Riemann Fellowship (Riemann Center for Geometry and Physics) (AK), FoPM, WINGS Program, the University of Tokyo (MO), and a Grant-in-Aid for Scientific Research on Innovative Areas 6003 (MO and TW), MEXT, Japan.

A Appendix: Additional notation and background

Some definitions not included (and notations not explained) in the main text:

[39] cf also [44] and [45].

String theorists do not necessarily have lot of experience with number fields or theory of complex multiplication. For those readers, the appendices[40] of [15] will be useful. Materials there include basics about number fields, and the definitions of *totally imaginary field, totally real field, CM field, the totally real subfield of a CM field, CM-type of a weight-1 rational Hodge structure, reflex field of a CM-type, primitivity of a CM-type.* The notations such as $\mathbb{Q}[x]$, $[E:F]$ and $\mathrm{Tr}_{E/F}$ of a field extension E/F as well as its properties are also explained there. So, we do not include those materials in this article. A *CM algebra* is the direct sum of a finite number of CM fields.

Notation A.1. $M_n(A)$ for an algebra A is the algebra of A-valued $n \times n$ matrices.

Definition A.2. An algebra D over a field F is a *division algebra* if any non-zero element $x \in D$ has an inverse x^{-1} with respect to the multiplication law of the algebra D.

Then $x^{-1} \cdot x = 1 = x \cdot x^{-1}$. A division algebra D is regarded as a field if the multiplication law of the algebra D is commutative (abelian).

Definition A.3. A finite dimensional algebra A over a field F is *semi-simple* if there is no non-zero nilpotent ideal.

A.4. Although a minimum explanation on rational (pure) Hodge structure is given already in Appendix B of [15], we also repeat some of it here without worrying about overlap. That is partly because we should have an eye on something beyond the most conventional pure rational Hodge structure in this article, and also because not much emphasis was given to the role played by a polarization of a rational Hodge structure in [15]. So, let us start from the basics.

Definition A.5. Let $V_\mathbb{Q}$ be a vector space over $\mathbb{Q}$. A *pure rational Hodge structure on $V_\mathbb{Q}$ of weight-m* is a decomposition of a vector space over $\mathbb{C}$,

$$V_\mathbb{Q} \otimes \mathbb{C} \cong \oplus_{p,q}^{(p+q=m)} [V_\mathbb{Q} \otimes \mathbb{C}]^{p,q} \tag{A.1}$$

satisfying $([V_\mathbb{Q} \otimes \mathbb{C}]^{p,q})^{\text{c.c.}} = [V_\mathbb{Q} \otimes \mathbb{C}]^{q,p}$. The word "pure" is often omitted; it is retained only when there is a high chance of confusion with a mixed rational Hodge structure (mentioned in footnote 11) or with a generalized Hodge structure we introduce in Def. 2.9.

[40] Some materials in the appendix of the preprint version are placed within the main text of the journal version.

See [15, App. B] or any math literatures for the definition of a *Hodge substructure, simple rational Hodge structure*, and *the level of a pure rational Hodge structure.*

A.6. For a rational pure Hodge structure of weight-m on a vector space $V_{\mathbb{Q}}$, the added data on top of the vector field $V_{\mathbb{Q}}$, i.e., the decomposition, can also be encoded by giving a representation

$$h : S^1 \longrightarrow \mathrm{GL}(V_{\mathbb{Q}} \otimes \mathbb{R}), \qquad h(e^{i\alpha})|_{[V_{\mathbb{Q}} \otimes \mathbb{C}]^{p,q}} = e^{-i\alpha(p-q)} \tag{A.2}$$

(the representation h cannot reproduce the information $m = p + q$, so the weight-m needs to be retained along with h).

Notation A.7. Let $(V_{\mathbb{Q}}, h)$ be a pure rational Hodge structure. Then

$$\begin{aligned} \mathrm{End}(V_{\mathbb{Q}})^{\mathrm{Hdg}} &:= \{\phi \in \mathrm{Hom}_{\mathbb{Q}}(V_{\mathbb{Q}}, V_{\mathbb{Q}}) \mid \phi([V \otimes \mathbb{C}]^{p,q}) \subset [V \otimes \mathbb{C}]^{p,q}\}, && \text{(A.3)} \\ &= \left\{\phi \in \mathrm{Hom}_{\mathbb{Q}}(V_{\mathbb{Q}}, V_{\mathbb{Q}}) \mid \phi \circ h(e^{i\alpha}) = h(e^{i\alpha}) \circ \phi\right\}. && \text{(A.4)} \end{aligned}$$

We call it *the endomorphism algebra*, and its elements *endomorphisms* in this article. We should refer to those elements as Hodge-structure-preserving endomorphisms of the vector space $V_{\mathbb{Q}}$ for a general $(V_{\mathbb{Q}}, h)$; when we deal with the 1st cohomology groups of an abelian variety, however, such Hodge-structure-preserving endomorphisms originate from the group-law preserving morphisms of an abelian variety to itself. So, for this reason, it is not too bad to use the word that does not sound right (certainly not right especially for various simple components of $H^{k>1}(T^{2n}; \mathbb{Q})$).

Definition A.8. A bilinear form $\mathcal{Q} : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to \mathbb{Q}$ on a vector space $V_{\mathbb{Q}}$, either symmetric (for even m) or anti-symmetric (for odd m), is said to be a *polarization of a pure rational Hodge structure* $(V_{\mathbb{Q}}, h)$ *of weight-m*, if $\mathcal{Q}(x, y) \in \mathbb{C}$ can be non-zero for $x \in [V \otimes \mathbb{C}]^{(p_1,q_1)}$ and $y \in [V \otimes \mathbb{C}]^{(p_2,q_2)}$ only when $p_1 = q_2$ and $q_1 = p_2$, and $i^{p-q}\mathcal{Q}(x, x^{\mathrm{cc}}) > 0$ for $x_{\neq 0} \in [V \otimes \mathbb{C}]^{p,q}$. A pure rational Hodge structure $(V_{\mathbb{Q}}, h)$ of weight-m is said to *be polarizable* when $(V_{\mathbb{Q}}, h)$ admits a polarization.

Let ψ be a polarization[41] of an abelian variety X of complex dimension n. Then the rational Hodge structures of weight m on $H^m(X; \mathbb{Q})$ admits a polarization $\mathcal{Q}_\psi$ given by $\mathcal{Q}_\psi(x, y) = \int_X \psi^{n-m} \wedge x \wedge y$.

[41] Its definition is found at the beginning of section 2.5.

Definition A.9. Let $(V_{\mathbb{Q}}, h)$ be a pure rational Hodge structure of weight-m. Its *Hodge group*, denoted by $\mathrm{Hg}((V_{\mathbb{Q}}, h))$ or $\mathrm{Hg}(h)$, is the minimal algebraic variety of $\mathrm{GL}(V_{\mathbb{Q}})$ with the group law from $\mathrm{GL}(V_{\mathbb{Q}})$ given by defining equations that involve only rational coefficients (i.e., in $\mathbb{Q}$, not in $\mathbb{C}$), so that all the points $h(e^{i\alpha})$ for $e^{i\alpha} \in S^1$ satisfy those defining equations.

Most of literatures referring to a Hodge group is for a polarizable pure rational Hodge structure. But it is possible to define such a notion for a rational Hodge structure that is not necessarily polarizable; whether such a $\mathrm{Hg}(h)$ still has a nice property is a separate question.

Representations of semi-simple algebras: we record a few basic known facts about representations of a finite-dimensional semi-simple algebras over $\mathbb{Q}$ for convenience of readers. Those facts are used in the main text.

Lemma A.10. This is known as Wedderburn's theorem. A finite dimensional semi-simple algebra $\mathfrak{R}$ over $\mathbb{Q}$ has a structure

$$\mathfrak{R} \cong \oplus_{\alpha \in \mathcal{A}} M_{n_\alpha}(D_\alpha) \tag{A.5}$$

for some finite set $\mathcal{A}$, $n_\alpha \in \mathbb{N}$, and a division algebra D_α.

When $\mathfrak{R}$ has a faithful representation on a vector space $V_{\mathbb{Q}}$ over $\mathbb{Q}$, $\dim_{\mathbb{Q}}(V_{\mathbb{Q}}) \geq \sum_\alpha n_\alpha q_\alpha^2 [k_\alpha : \mathbb{Q}]$, where k_α is the center of D_α, and q_α is the positive integer such that $[D_\alpha : k_\alpha] = q_\alpha^2$.

The following facts (Lemmas A.11, A.12) are regarded so trivial by mathematicians that we have to read that out between the lines in textbooks on semi-simple algebras. The authors are unable to refer to a specific text for this reason. For the reader with a background in string theory, it will still be better that they are written down explicitly.[42]

Lemma A.11. Let F be a number field (with $[F : \mathbb{Q}] < \infty$), and $\{\tau_{a=1,\cdots,[F:\mathbb{Q}]}\}$ its embeddings to $\overline{\mathbb{Q}} \subset \mathbb{C}$. Let $V_{\mathbb{Q}}$ be a vector space over $\mathbb{Q}$. We think of only the cases that $[F : \mathbb{Q}] = \dim_{\mathbb{Q}} V_{\mathbb{Q}}$ here.

Suppose that F acts non-trivially on $V_{\mathbb{Q}}$ (with $\mathbb{Q} \subset F$ acting as the scalar multiplication on the vector space $V_{\mathbb{Q}}$). The action of F on $V_{\mathbb{Q}}$ can be diagonalized simultaneously; the $\dim_{\mathbb{Q}} V$ eigenvectors can be chosen in a following way. First, choose any non-zero element $v_* \in V_{\mathbb{Q}}$, and arbitrary basis $\{\omega_{i=1,\cdots,[F:\mathbb{Q}]}\}$ of the vector space $F/\mathbb{Q}$. Then $\{\omega_i \cdot v_*\}_{i=1,\cdots,[F:\mathbb{Q}]}$

[42]The appendix B.2 of the preprint version of [15] (main text II.B.3 of the journal version) has a little more pedagogical explanation on the first half of Lemma A.11. The statement here is slightly polished up from the version there, however.

can be used as a basis of the vector space $V_{\mathbb{Q}}$ over $\mathbb{Q}$. Now, we can choose the eigenvectors to be[43]

$$v_a := \sum_i (\omega_i \cdot v_*)\tau_a(\eta_i), \qquad a = 1, \cdots, [F:\mathbb{Q}], \tag{A.6}$$

where $\{\eta_{j=1,\cdots,[F:\mathbb{Q}]}\}$ is the basis of $F/\mathbb{Q}$ dual to $\{\omega_i\}$ with respect to the bilinear form $\mathrm{Tr}_{F/\mathbb{Q}}[xy]$. That is, $\mathrm{Tr}_{F/\mathbb{Q}}[\omega_i\eta_j] = \delta_{ij}$. The matrix $(\tau_a(\eta_i))_{ai}$ is used as the inverse matrix of $(\tau_a(\omega_j))_{ja}$. For any element $x \in F$, its action on $V_{\mathbb{Q}} \otimes \mathbb{C}$ is given by $x \cdot v_a = v_a\tau_a(x)$, i.e., the eigenvalue of $x\cdot$ is $\tau_a(x)$ for the eigenvector v_a. All those eigenvectors v_a $(a = 1, \cdots, [F:\mathbb{Q}])$ are obtained from one of them, say, v_{a*}, by applying Galois transformations on the coefficients $\tau_{a*}(\eta_i)$ of the expansion of v_{a*} with respect to the rational basis $\{(\omega_i \cdot v_*)_{i=1,\cdots,[F:\mathbb{Q}]}\}$, because $\tau_a = \sigma_a \cdot \tau_{a*}$ for some $\sigma_a \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We may express this in the form of $v_a = v_{a*}^{\sigma_a}$.

For any basis $\{\eta'_{i=1,\cdots,[F:\mathbb{Q}]}\}$ of $F/\mathbb{Q}$, there exists a basis $\{v'_i\}$ of $V_{\mathbb{Q}}$ where the simultaneous eigenvectors are in the form of $v_a = v'_i\tau_a(\eta'_i)$. To see this, just find the rational coefficient matrix $\eta_i = C_{ij}\eta'_j$ and set $v'_j := (\omega_i \cdot v_*)C_{ij}$.

Lemma A.12. Conversely, for any basis $\{\eta_j\}$ of $F/\mathbb{Q}$ and $\{v_i\}$ of V, one may construct a non-trivial action of F on the vector space V over $\mathbb{Q}$ so that $v_a := \sum_i v_i\tau_a(\eta_i)$ for $a = 1, \cdots, [F:\mathbb{Q}]$ are all eigenvectors of the action of F. The action of $x \in F$ on V claimed here is given as follows. First, write down the multiplication law in F as follows:

$$(x\cdot): \quad \omega_i \longmapsto \omega_k[A(x)]_{ki}, \tag{A.7}$$

where $\{\omega_i\}$ is the basis of $[F:\mathbb{Q}]$ dual to $\{\eta_j\}$, and $[A(x)]$ is a $\mathbb{Q}$-valued $[F:\mathbb{Q}] \times [F:\mathbb{Q}]$ matrix. Using this matrix, the action of x is

$$x\cdot: \quad v_i \longmapsto v_k[A(x)]_{ki}. \tag{A.8}$$

□

The facts above in both ways (Lemmas A.11 and A.12) hold for a general number field F not necessarily a CM field; the eigenspace decomposition does not have to be relevant to Hodge components.

[43] So, the action of F on $V_{\mathbb{Q}}$ splits into 1-dimensions on $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$ whenever $k \subset \overline{\mathbb{Q}}$ contains the normal closure of F in $\overline{\mathbb{Q}}$.

References

[1] G. W. Moore and N. Seiberg, "LECTURES ON RCFT," in *1989 Banff NATO ASI: Physics, Geometry and Topology.* 9, 1989.

[2] S. Gukov and C. Vafa, "Rational conformal field theories and complex multiplication," *Commun. Math. Phys.* **246** (2004) 181–210, [hep-th/0203213].

[3] G. W. Moore, "Arithmetic and attractors," hep-th/9807087.

[4] G. W. Moore, "Attractors and arithmetic," hep-th/9807056.

[5] Y. Taniyama and G. Shimura, *Complex multiplication of abelian varieties and its applications to number theory.* Mathematical Society of Japan, 1961. This book has been incorporated as a part of the book [6].

[6] G. Shimura, *Abelian Varieties with Complex Multiplication and Modular Functions.* Princeton university press, 1998.

[7] I. Piatetski-Shapiro and I. Shafarevich, "The arithmetic of K3 surfaces," in *Proc. Steklov Inst. Math*, vol. 132, pp. 45–57. 1973.

[8] J. Rizov, "Complex multiplication for K3 surfaces." math/0508018.

[9] S. Kondo and T. Watari, "String-theory Realization of Modular Forms for Elliptic Curves with Complex Multiplication," *Commun. Math. Phys.* **367** (2019), no. 1, 89–126, [arXiv:1801.07464].

[10] S. Kondo and T. Watari, "Modular parametrization as Polyakov path integral: cases with CM elliptic curves as target spaces," *Commun. Num. Theor. Phys.* **16** (2022), no. 2, 353–400, [arXiv:1912.13294].

[11] F. Denef, "Les Houches Lectures on Constructing String Vacua," in *Les Houches School of Physics: String Theory and the Real World: From Particle Physics to Astrophysics*, pp. 483–610. 2008. [arXiv:0803.1194].

[12] O. DeWolfe, A. Giryavets, S. Kachru, and W. Taylor, "Enumerating flux vacua with enhanced symmetries," *JHEP* **02** (2005) 037, [hep-th/0411061].

[13] P. S. Aspinwall and R. Kallosh, "Fixing all moduli for M-theory on K3xK3," *JHEP* **10** (2005) 001, [hep-th/0506014].

[14] G. W. Moore, "Strings and Arithmetic," in *Les Houches School of Physics: Frontiers in Number Theory, Physics and Geometry*, pp. 303–359. 2007. [hep-th/0401049].

[15] K. Kanno and T. Watari, "Revisiting arithmetic solutions to the $W = 0$ condition," *Phys. Rev. D* **96** (2017), no. 10, 106001, [arXiv:1705.05110].

[16] J. A. Harvey, G. W. Moore, and C. Vafa, "Quasicrystalline Compactification," *Nucl. Phys. B* **304** (1988) 269–290.

[17] K. Wendland, *Moduli spaces of unitary conformal field theories.* PhD thesis, Universität Bonn, 2000.

[18] S. Hosono, B. H. Lian, K. Oguiso, and S.-T. Yau, "Classification of c = 2 rational conformal field theories via the Gauss product," *Commun. Math. Phys.* **241** (2003) 245–286, [hep-th/0211230].

[19] M. Chen, "Complex multiplication, rationality and mirror symmetry for Abelian varieties," *J. Geom. Phys.* **58** (2008) 633–653, [math/0512470].

[20] A. Kapustin and D. Orlov, "Vertex algebras, mirror symmetry, and D-branes: The Case of complex tori," *Commun. Math. Phys.* **233** (2003) 79–136, [hep-th/0010293].

[21] Meng Chen, *Complex Multiplication, Rationality and Mirror Symmetry for Abelian Varieties and K3 Surfaces.* PhD thesis, Universität Bonn, 2007.

[22] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves.* No. 88 in Mathematical Surveys and Monographs. American Mathematical Soc., 2004.

[23] C. Van Enckevort, "Moduli spaces and D-brane categories of tori using SCFT." hep-th/0302226.

[24] T. Shioda, "Geometry of fermat varieties," in *Number Theory Related to Fermat's Last Theorem*, pp. 45–56. Springer, 1982.

[25] C. Borcea, "Calabi-yau threefolds and complex multiplication," in *Mirror Symmetry I*, vol. 9 of *AMS/IP Studies in Advanced Mathematics*, pp. 431–443. International Press, 1998.

[26] C. Voisin, *Miroirs et involutions sur les surfaces K3*. Institut des Hautes Etudes Scientifiques [IHES], 1993.

[27] A. P. Braun and T. Watari, "The Vertical, the Horizontal and the Rest: anatomy of the middle cohomology of Calabi-Yau fourfolds and F-theory applications," *JHEP* **01** (2015) 047, [arXiv:1408.6167].

[28] V. Golyshev, V. Lunts, and D. Orlov, "Mirror symmetry for abelian varieties," *J. Alg. Geom.* **10** (2001), no. 3, 433–496, [math/9812003].

[29] N. Hitchin, "Generalized Calabi-Yau manifolds," *Quart. J. Math.* **54** (2003) 281–308, [math/0209099].

[30] M. Gualtieri, "Generalized complex geometry," *Ann. Math. (2)* **174** (2011), no. 1, 75–123.

[31] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[32] C. A. Peters and J. H. Steenbrink, *Mixed hodge structures*, vol. 52 of *Ergenibsse der Mathematik und ...* Springer, 2008.

[33] C. Birkenhake and H. Lange, *Complex abelian varieties*, vol. 302 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2004.

[34] C. Birkenhake and H. Lange, *Complex tori*, vol. 177 of *Progress in Mathematics*. Springer, 1999.

[35] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, vol. 13 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second ed., 1992.

[36] H. Nagao and Y. Tsushima, *Representations of finite groups*. Academic Press, Inc., Boston, MA, 1989. Translated from the Japanese.

[37] J.S.Milne, "Complex multiplication." online lecture note version as of 19th May 2022 https://www.jmilne.org/math/CourseNotes/CM.pdf.

[38] C. Rohde, *Cyclic coverings, Calabi-Yau manifolds and complex multiplication*, vol. 1975 of *Lecture Notes in Mathematics*. Springer, 2009.

[39] P. S. Aspinwall, "K3 surfaces and string duality," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 421–540. 11, 1996. [hep-th/9611137].

[40] T. Shioda and N. Mitani, "Singular abelian surfaces and binary quadratic forms," in *Classification of algebraic varieties and compact complex manifolds*, pp. 259–287. Springer, 1974.

[41] Y. André, *G-functions and geometry.* Springer, 1989.

[42] F. Oort, "Canonical liftings and dense sets of CM-points," *Arithmetic geometry (Cortona, 1994)* **37** (1997) 228–234.

[43] Y. André, "Distribution des points CM sur les sous-variétés des variétés de modules de variétés abéliennes," *Jussieu prépublication* **120** (1997).

[44] J. Tsimerman, "A proof of the Andre-Oort conjecture for A_g." arXiv:1506.01466.

[45] B. Moonen and F. Oort, "The Torelli locus and special subvarieties," in *Handbook of moduli. Vol. II*, vol. 25 of *Adv. Lect. Math. (ALM)*, pp. 549–594. Int. Press, Somerville, MA, 2013. [arXiv:1112.0933].