Genus 1-fibered CY 3-folds with 5-sections – A GLSM perspective –

[with E. Scheidegger, T. Schimannek: 2107.05647[hep-th]]

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Outline

Motivation & Summary

Non-abelian GLSMs

Genus 1 fibrations
- $1D$ Rødland model
- GLSM 1
- GLSM 2

Conclusions

F-theory

- F-theory needs elliptically/genus 1-fibered Calabi-Yaus.

  [Vafa 96][tons of references!, see eg. recent TASI lectures by Cvetic-Lin, Weigand 18]

- N-sections [Braun-Morrison 14][many others]
  - An N-section intersects the generic fibre N times.
  - This leads to a $\mathbb{Z}_N$-symmetry in F-theory.
- $N \leq 4$ can be realised in terms of hypersurfaces or complete intersections in toric ambient spaces.
  - What about $N > 4$?
  - Constructions outside the realm of toric geometry.

Topological Strings – I

- Modular bootstrap:

  [Huang-Katz-Klemm 15][dZ-G-H-KP-K-L 17][Lee-Lerche-Weigand 18][Cota-Klemm-Schimannek 19]

  - All-genus Gopakumar-Vafa invariants thanks to modular properties of topological string partition function.
- Conjecture: For genus 1-fibered CYs with N-sections

$$Z_{\text{top}}(\tau, t, \lambda) = Z_0(\tau, \lambda)\left(1 + \sum_{\beta \in H_2(B,\mathbb{Z})} Z_\beta(\tau, \lambda) e^{2\pi i \beta_j t^j}\right)$$

$$Z_{\beta>0}(\tau, \lambda) = \frac{\phi_\beta(\tau, \lambda)}{\eta(N\tau)^{12\beta\cdot K_B} \prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} \phi_{-2,1}(N\tau, s\lambda)}$$

  - $\phi_\beta, \phi_{-2,1} \ldots \Gamma_1(N)$ Jacobi forms

Topological Strings – II

- GV spectroscopy [Oehlmann-Schimannek 19]
  - Gopakumar-Vafa invariants encode mathematical properties of the genus 1 fibration.
  - Useful tool to compute the F-theory spectrum.
- M-/F-theory dualities
- Extremal transitions between toric and non-toric CYs through Higgs transitions in M-/F-theory.
  - F-theory with charge 5 hypermultiplets
- Classification of 5-section geometries via extremal transitions.
  - List of 23
- Moduli space and monodromies

  [Schimannek 19][Cota-Klemm-Schimannek 19][Schimannek 21]
  - $\Gamma_0(5)$ and $\Gamma_1(5)$
  - Tate-Shafarevich group
  - Homological projective duality

Mathematics

- Systematic construction of a large class of new Calabi-Yau geometries
  - Complete intersections in Grassmannian fibrations
  - Pfaffian Calabi-Yaus
  - Also works beyond genus 1-fibrations
- Compute topolgical characteristics of these Calabi-Yaus
  - Requires heavy mathematical machinery.
  - For instance: spectral sequences, Schubert calculus, Schur functors, Borel-Weil-Bott, ...
- The constructions naturally fit into the framework of homological projective duality. [Kuznetsov 05]
  - Lots of new conjectures for homologically projective dual Calabi-Yaus and derived equivalences.

GLSMs

- Non-abelian GLSMs can be used to construct new Calabi-Yaus beyond toric geometry. [Hori-Tong 06][J-K-L-M-R 12] [Hori-JK 13,16][Gerhardus-Jockers 15,16][Sharpe-JK 18,19]
  - CYs als vacuum manifolds of the GLSM. [Witten 93]
  - Phases: Different CYs in different regions of parameter space.
- New dualities:
  - Hori duality [Hori 11]
  - Non-abelian mirror symmetry [Gu-Sharpe 18]
- Categories associated to phases of GLSMs are (conjecturally) derived equivalent.
- One can realise elliptic curves via non-abelian GLSMs. [Hori-JK 13]
  - Can we construct genus 1 fibrations using non-abelian GLSMs?

GLSM data – I

- G ... a compact Lie group (gauge group)
- V ... space of chiral fields $\phi_i \in V$
- $\rho_V : G \to GL(V)$ ... faithful complex representation
  - CY condition: $G \to SL(V)$
- $R : U(1)_V \to GL(V)$ ... vector R-symmetry
  - R_i ... R-charges
- $T \subset G$ ... maximal torus
  - Lie algebras: $\mathfrak{g} = Lie(G)$, $\mathfrak{t} = Lie(T)$
  - $Q_i^a \in \mathfrak{t}_{\mathbb{C}}^*$ ... gauge charges of chiral fields

GLSM Data – II

- $t^a \in \mathfrak{g}^*_{\mathbb{C}}$ ... FI-theta parameters
  - $t^a = \zeta^a - i\theta^a$ $\quad\quad$ ζ^a : real, $\quad$ θ^a : 2π-periodic
  - $t^a \leftrightarrow$ Kähler moduli of the CY
- $\sigma_a \in \mathfrak{g}_{\mathbb{C}}$ ... scalar components of the vector multiplet
- $W \in \mathrm{Sym}\, V^*$ ... superpotential
  - G-invariant
  - R-charge 2
  - non-zero for compact CYs
- Classical potential

$$U = \frac{1}{8e^2}|[\sigma, \bar{\sigma}]|^2 + \frac{1}{2}\left(|Q(\sigma)\phi|^2 + |Q(\bar{\sigma})\phi|^2\right) + \frac{e^2}{2}D^2 + |F|^2$$

Phases of the GLSM

- Higgs branch: $\sigma = 0$
- D-terms

$$\mu(\phi) = \zeta$$

  - $\mu : V \to \mathfrak{g}^*$ ... moment map
  - G is broken to a subgroup
- F-terms

$$dW = 0$$

- Phases: parameter space gets divided into chambers
- Classical vacua

$$X_\zeta = \{dW^{-1}(0)\} \cap \mu^{-1}(\zeta)/G$$

Phase boundaries

- Coulomb branch: $\phi = 0$, $\zeta = 0$
- σ fields can take any value classically.
- One-loop corrections generate an effective potential.

$$\widetilde{W}_{eff} = -t(\sigma) - \sum_i Q_i(\sigma)(\log(Q_i(\sigma)) - 1) + \pi i \sum_{\alpha>0} \alpha(\sigma)$$

  - $\alpha > 0$: positive roots
- Lifted except for points, lines, etc. $\rightarrow$ discriminant
  - One can smoothly interpolate between the phases.
  - The CY undergoes a topological transition.
- Mixed Branches if $\mathrm{rk}\, T > 1$.

Sphere partition function

- Sphere partition function

[Benini-Cremonesi 12][Doroud-Gomis-LeFloch-Lee 12]

$$Z_{S^2} = C \sum_{m\in\mathbb{Z}^{\mathrm{rk}G}} \int_\gamma d^{\mathrm{rk}G}\sigma \prod_{\alpha>0}\left(\frac{\alpha(m)^2}{4}+\alpha(\sigma)^2\right) \cdot \prod_{j=1}^{\dim V} \frac{\Gamma\left(iQ_j(\sigma)-\frac{Q_j(m)}{2}+\frac{R_j}{2}\right)}{\Gamma\left(1-iQ_j(\sigma)-\frac{Q_j(m)}{2}-\frac{R_j}{2}\right)} e^{-4\pi i\zeta(\sigma)-i\theta(m)}$$

- Z_{S^2} computes the exact Kähler potential on $\mathcal{M}_K$.

[Jockers-Kumar-Lapan-Morrison-Romo 12][Gomis-Lee 12]

[Gerchkovitz-Gomis-Komargodski 14][Gomis-Hsin-Komargodski-Schwimmer-Seiberg-Theisen 15]

  - Periods and Picard-Fuchs operators.

An elliptic curve – I

- Consider a GLSM with gauge group $G = U(2)$ and the following field content: [Hori-JK 13]

	$p^1, \ldots, p^5$	$x_1^a, \ldots, x_5^a$	FI
$U(2)$	$\det^{-1}$	$\square$	ζ

- Superpotential:

$$W = \sum_{i,j=1}^{5} \sum_{k=1}^{5} A_k^{ij} p^k \sum_{a,b=1}^{2} \varepsilon_{ab} x_i^a x_j^b = \sum_{i,j=1}^{5} A^{ij}(p)[x_i x_j]$$

- D-term:

$$(xx^\dagger)_a^b - \left(\sum_{k=1}^{5} |p^k|^2 \right) \mathrm{id}_a^b = \zeta \mathrm{id}_a^b.$$

An elliptic curve – II

- $\zeta \gg 0$: Grassmannian phase:

$$X_0 = \{[x_i x_j] \in G(2,5) | A^{ij}_k [x_i x_j] = 0\}$$

- $\zeta \ll 0$: Pfaffian Phase (strongly coupled):

$$Y_0 = \{p^1 \dots p^5 \in \mathbb{P}^4 | \mathrm{rk} A^{ij}(p) = 2\}$$

- Coulomb branch:

$$e^{-t} = -\frac{1}{2}(11 \pm 5\sqrt{5})$$

- X_0 and Y_0 describe the same elliptic curve.

An elliptic curve – III

- Fundamental period (via sphere partition function)

$$\varpi_0 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} z^n \qquad z = e^{-t}$$

  - Series coefficients are the Apéry numbers.
  - Given this, we can construct the Picard-Fuchs operator.

- Inverse of the mirror map: [OEIS]

$$q \prod_{n=1}^{\infty} (1-q^n)^{5\left(\frac{5}{n}\right)} \in \Gamma_1(5), \qquad q = e^{2\pi i \frac{\varpi_1}{\varpi_0}}.$$

  - $\left(\frac{m}{n}\right)$ ... Jacobi symbol

Constructing genus 1 fibrations

- Construct genus 1 fibered Calabi-Yau threefolds using X_0, Y_0.
- Focus on simplest examples:
  - Two-parameter models
  - Complete intersections/Pfaffians in fibrations over $\mathbb{P}^2$
- Ideally: Trivial fibration of X_0, Y_0 over $\mathbb{P}^2$
  - Works, but not in a single GLSM.
- Consider two examples of GLSMs with gauge group $G = U(1) \times U(2)$.

GLSM 1 – I

- Different constructions of these geometries exist.

[Inoue 19][Sharpe-JK 19][Prince 19]

- Field content:

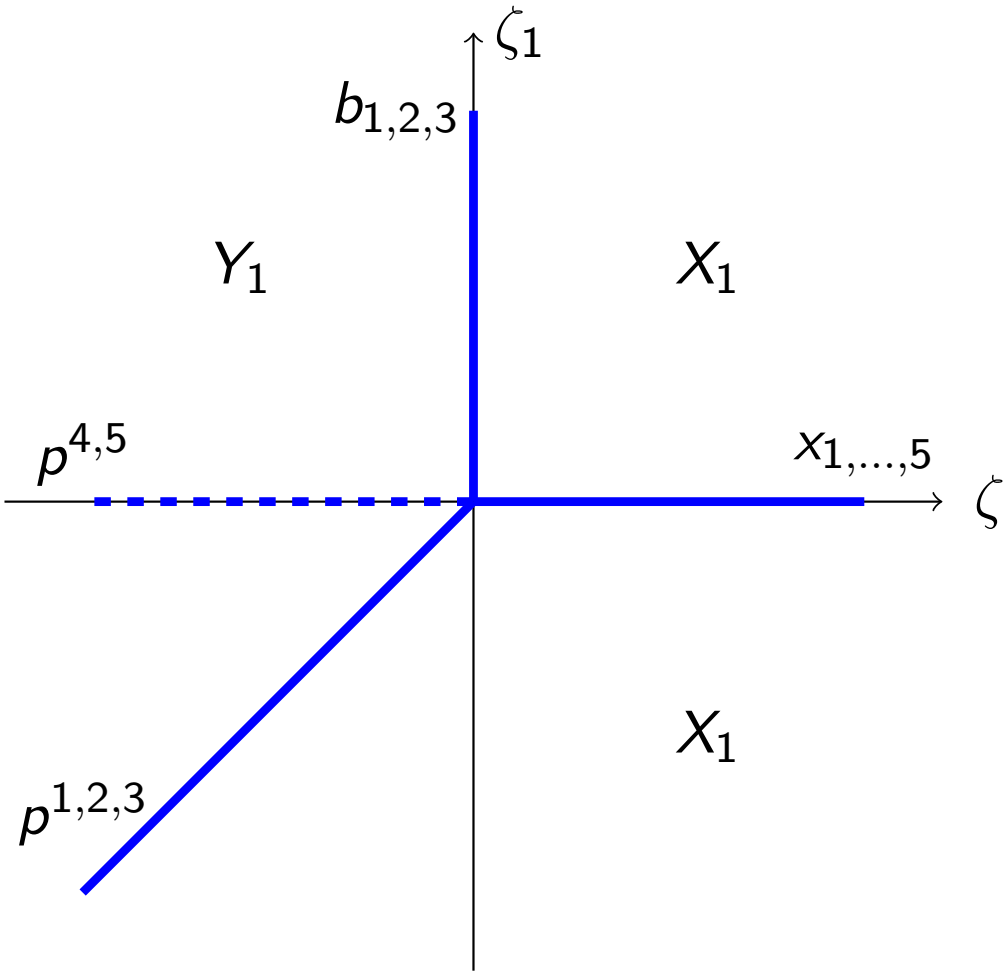
	p^1, p^2, p^3	p^4, p^5	$x_1^a, \ldots, x_5^a$	b_1, b_2, b_3	FI
$U(2)$	$\det^{-1}$	$\det^{-1}$	$\square$	1	ζ
$U(1)$	-1	0	0	1	ζ_1

- Superpotential:

$$W = \sum_{i,j=1}^{5} \underbrace{\left[\sum_{k,l=1}^{3} A_k^{ijl} b_l p^k + \sum_{k=4}^{5} A_k^{ij} p^k \right]}_{A^{ij}(p,b)} [x_i x_j]$$

GLSM 1 – II

- Classical phase diagram:



GLSM 1 – III

- Grassmannian phase X_1 (weakly coupled):

$$X_1 : \quad \sum_{i,j=1}^{5} \sum_{l=1}^{3} A_k^{ijl} b_l [x_i x_j] = 0, \quad k = 1, 2, 3$$

$$\sum_{i,j=1}^{5} A_k^{ij} [x_i x_j] = 0, \quad k = 4, 5.$$

  - Codimension 5 complete intersection in $G(2,5) \times \mathbb{P}^2$
- "Other" X_1-phase:

$$b_{1,2,3} \leftrightarrow p^{1,2,3}$$

GLSM 1 – IV

- Pfaffian phase Y_1 (strongly coupled)

$$Y_1 : \quad \{(p,b) | \mathrm{rk} A_{ij}(p,b) = 2\}$$

  - Confirmed via Hori-dual
  - Non-trivial fibration over $\mathbb{P}^2$
  - Extra phase boundary lifted by F-terms.
- Topological data:

	$h^{1,1}$	$h^{1,2}$	$c_2 \cdot J_1$	$c_2 \cdot J_2$	J_1^3	$J_1^2 J_2$	$J_1 J_2^2$	J_2^3
X_1	2	47	66	36	15	15	5	0
Y_1	2	47	36	36	0	5	5	0

GLSM 1 – V

- Coulomb branch:

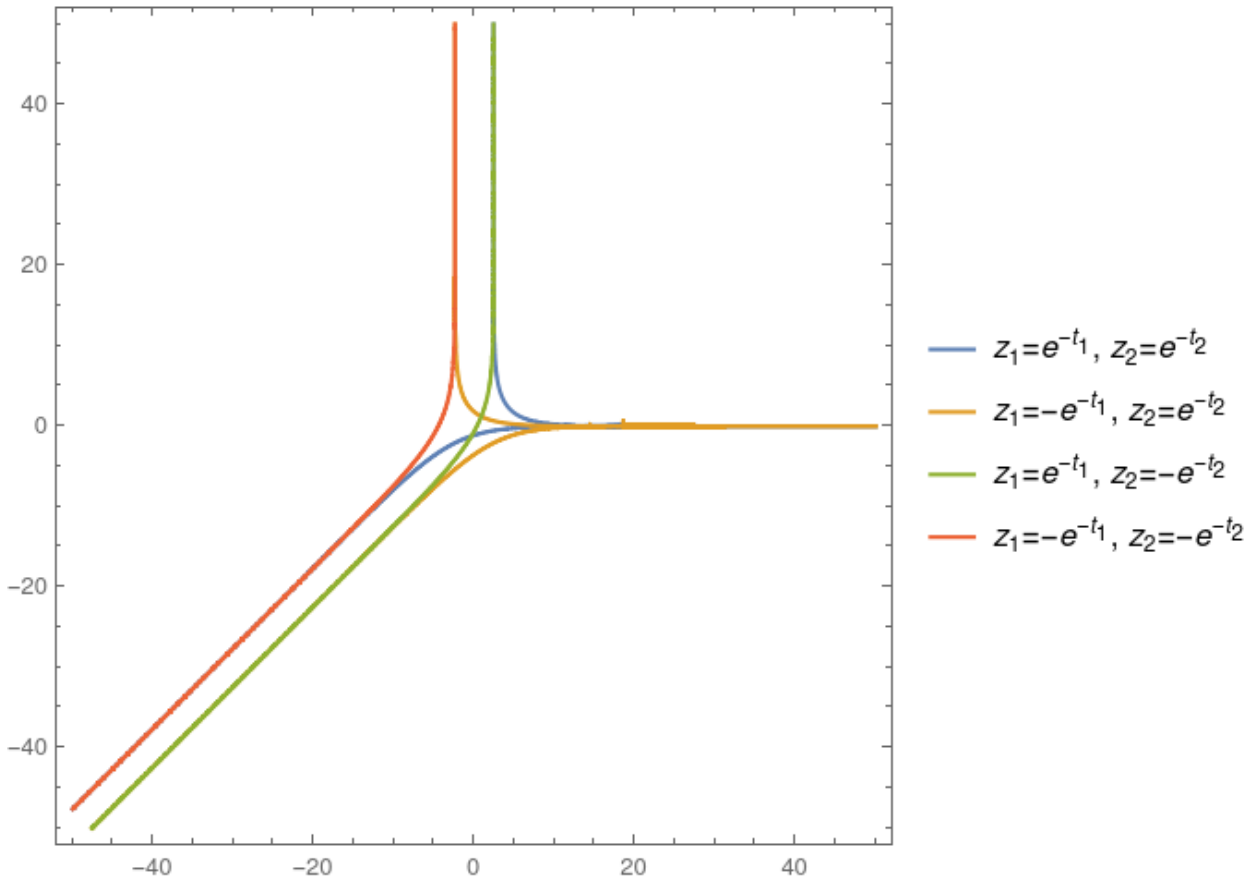
$$e^{-t} = \frac{1}{(1+z)^2(1+z+w)^3}, \quad e^{-t_1} = -\frac{1}{\left(1+\frac{1}{w}+\frac{z}{w}\right)^3},$$

$$z^5 = 1, \qquad z = \frac{\sigma_2}{\sigma_1}, \quad w = \frac{\sigma_3}{\sigma_1}$$

  - $\sigma_1, \sigma_2 \in \mathfrak{t}_{U(2)}$, $\sigma_3 \in \mathfrak{t}_{U(1)}$
- Coincides with the "mirror calculation".
  - Fundamental period ϖ_0 via sphere partition function.
  - Picard-Fuchs operators such that $\mathcal{L}\varpi_0 = 0$.

GLSM 1 – VI

- Quantum phase diagram:



GLSM 2 – I

- Related GLSM and mathematics constructions.

[Caldararu 00][Donagi-Sharpe 07]

- Field content:

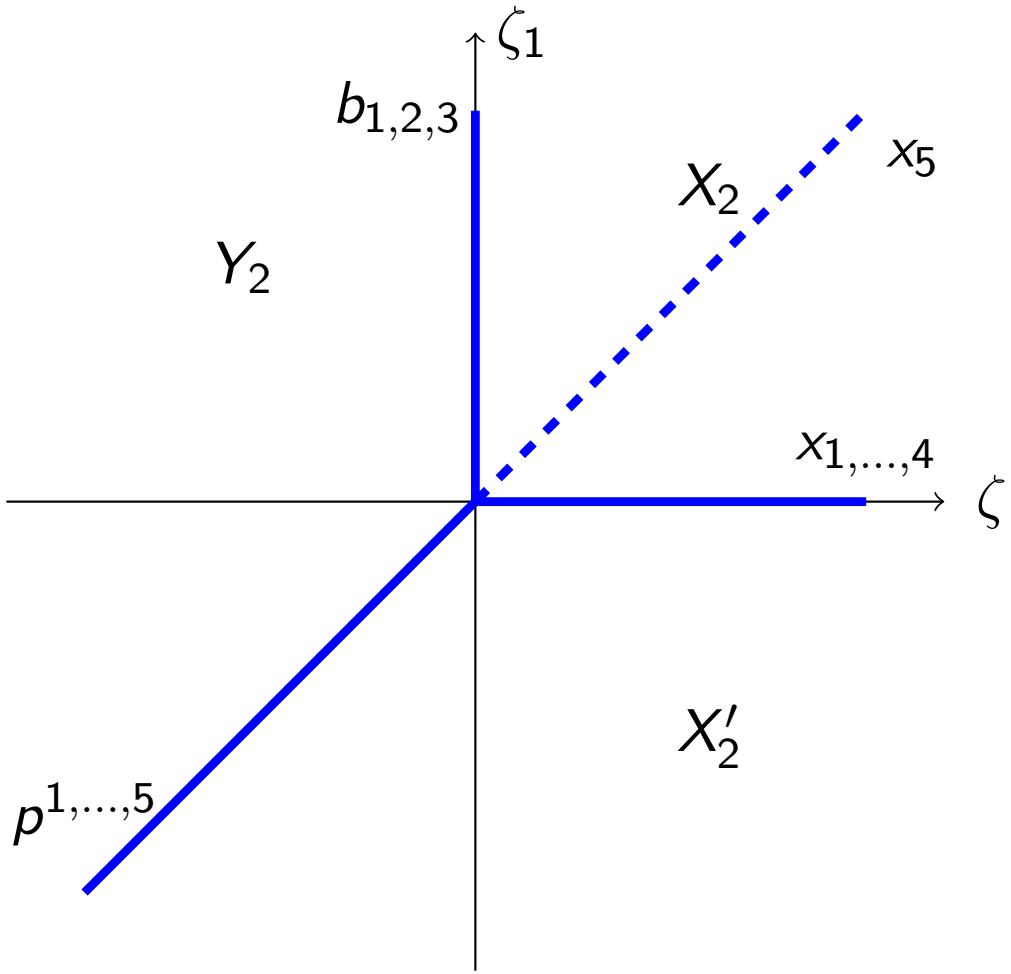
	$p^1, \ldots, p^5$	$x_1^a, \ldots, x_4^a$	x_5^a	b_1, b_2, b_3	FI
$U(2)$	$\det^{-1}$	$\square$	$\square$	1	ζ
$U(1)$	-1	0	1	1	ζ_1

- Superpotential:

$$W = \sum_{i,j=1}^{4} \sum_{k=1}^{5} \sum_{l=1}^{3} A_k^{ijl} b_l p^k [x_i x_j] + \sum_{i=1}^{4} \sum_{k=1}^{5} A_k^{i5} p^k [x_i x_5].$$

GLSM 2 – II

- Classical phase diagram:



GLSM 2 – III

- Grassmannian phase X_2 (weakly coupled):

$$X_2: \quad \sum_{i,j=1}^{4}\sum_{l=1}^{3} A_k^{ijl} b_l [x_i x_j] + \sum_{i=1}^{4} A_k^{i5}[x_i x_5] = 0, \qquad k = 1, \ldots, 5$$

- Phase X_2' (weakly coupled):

$$X_2': \quad \sum_{i,j=1}^{4}\sum_{k=1}^{5} A_k^{ijl} p^k [x_i x_j] = 0, \qquad l = 1, 2, 3$$

$$\sum_{i=1}^{4}\sum_{k=1}^{5} A_k^{i5} p^k x_{i,a} = 0, \qquad a = 1, 2$$

  - Complete intersection in a $G(2,4)$-fibration over $\mathbb{P}^4$.
  - Not a genus 1-fibration over $\mathbb{P}^2$

GLSM 2 – IV

- Pfaffian phase Y_2 (strongly coupled):

$$Y_1 : \quad \{(p,b)|\mathrm{rk}A_{ij}(p,b) = 2\}$$

- Topological data:

	$h^{1,1}$	$h^{1,2}$	$c_2 \cdot J_1$	$c_2 \cdot J_2$	J_1^3	$J_1^2 J_2$	$J_1 J_2^2$	J_2^3
X_2	2	49	56	36	8	11	5	0
Y_2	2	49	50	36	5	9	5	0
X_2'	2	49	56	50	8	13	11	5

Conclusions

- With the help of non-abelian GLSMs have constructed (simple) examples of genus 1 fibered Calabi-Yau threefolds with 5-sections.
- Apart from the GLSMs themselves, one can do a lot of interesting things with them.
  - Algebraic geometry and topology
  - Topological strings
  - M-/F-theory
  - Extremal transitions
  - (Conjectured) derived equivalences
  - Modularity
  - Classification

Outlook

- There should be a similar story for 6-section geometries.

  [Hori-JK 13][in progress]

- More "non-abelian" elliptic curves? [Zagier,Almqvist-Zudilin 04]
- Systematic construction of a large number of new Calabi-Yaus.
- Mathematical proofs of the conjectured dualities and equivalences between Calabi-Yaus in the phases of GLSMs.
- Fourfolds.
- Interesting phenomenology?