

The Ratios Conjecture and negative moments of L-functions

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$\zeta(s)$ and $L(s, \chi)$

For $\Re s > 1$,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}.$$

- It has a meromorphic continuation to \mathbb{C} with a simple pole at $s = 1$.
- It satisfies a functional equation $\zeta(s) \leftrightarrow \zeta(1 - s)$
- Trivial zeros at $s = -2m, m \geq 1$.
- **(RH)** The non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.

$\zeta(s)$ and $L(s, \chi)$

A **Dirichlet character** is a completely multiplicative function

$$\chi : (\mathbb{Z}/d\mathbb{Z})^* \rightarrow \mathbb{C}^*$$

extended to \mathbb{Z} by periodicity, with the property that

$$\chi(a) = 0 \text{ whenever } (a, d) \neq 1.$$

χ is **primitive with conductor d** if d is the smallest modulus for which χ is a character modulo d .

For $\Re(s) > 1$ and χ a character (mod d),

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_{p \nmid d} \left(1 - \chi(p)p^{-s}\right)^{-1}.$$

- It has an analytic continuation to \mathbb{C}
- It satisfies a functional equation $L(s, \chi) \leftrightarrow L(1-s, \bar{\chi})$
- **(GRH)** The non-trivial zeros of $L(s, \chi)$ lie on the line $\Re(s) = 1/2$.

Conjecture (Montgomery)

The pair correlation of zeros of $\zeta(s)$ is the same as the pair correlation of matrices from the GUE ensemble.

- Katz and Sarnak looked at low-lying zeros in families of L -functions over function fields and showed that they follow the laws governed by the corresponding scaling limit of the symmetry group of the family.
- Can model L -functions in families by random matrix ensembles

$$\zeta(s) \iff \text{unitary ensemble}$$

$$L(s, \chi_d), \chi_d^2 = \chi_0 \iff \text{symplectic ensemble}$$

$$L(s, E \otimes \chi_d), \chi_d^2 = \chi_0 \iff \text{orthogonal ensemble}$$

Moments of L -functions

- **(RH)** The non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.
- **(Lindelöf hypothesis)** $|\zeta(1/2 + it)| = O(t^\epsilon), \forall \epsilon > 0$.
- Hardy and Littlewood (1916): moments of $\zeta(s)$

$$I_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} dt$$

Lindelöf hypothesis $\iff I_k(T) \ll T^{1+\epsilon}, k = 1, 2, \dots$

Conjecture (Keating, Snaith; Conrey, Farmer, Keating, Rubinstein, Snaith)

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt \sim M_k T(\log T)^{k^2}.$$

- $k = 1$: Hardy, Littlewood (1916)
- $k = 2$: Ingham (1932), Heath-Brown (1979)

Moments of L -functions

For a prime p , let

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a = \square \pmod{p}, (a, p) = 1 \\ -1 & \text{if } a \neq \square \pmod{p}, (a, p) = 1 \\ 0 & \text{if } p|a. \end{cases}$$

Extend multiplicatively. Let

$$\chi_d(n) = \left(\frac{d}{n}\right).$$

Conjecture (Keating, Snaith; Conrey, Farmer, Keating, Rubinstein, Snaith)

$$\sum_{0 < d \leq X}^* L\left(\frac{1}{2}, \chi_d\right)^k \sim C_k X (\log X)^{\frac{k(k+1)}{2}}.$$

Conjecture (Keating, Snaith; Conrey, Farmer, Keating, Rubinstein, Snaith)

$$\sum_{0 < d \leq X}^* L\left(\frac{1}{2}, \chi_d\right)^k = XP_k(\log X) + O(X^{1-\delta}),$$

where $\deg(P_k) = k(k+1)/2$, $\delta > 0$.

- $k = 1, 2$: Jutila (1981)
- $k = 2, 3$: Soundararajan (2000)
- $k = 3$: Diaconu, Goldfeld, Hoffstein (2003)
- $k = 4$: Shen (2019) on GRH, using ideas of Soundararajan and Young
- Upper bounds of the right order of magnitude, on GRH:
Soundararajan, Harper
- Lower bounds of the right order of magnitude: Soundararajan, Rudnick

The Ratios Conjecture for $\zeta(s)$

Conjecture (Farmer, 1993)

For $s = 1/2 + it$ and complex numbers $\alpha, \beta, \gamma, \delta$ of size $c/\log T$, such that $\Re\alpha, \Re\beta, \Re\gamma, \Re\delta > 0$ we have

$$\frac{1}{T} \int_0^T \frac{\zeta(s + \alpha)\zeta(1 - s + \beta)}{\zeta(s + \gamma)\zeta(1 - s + \delta)} dt \sim 1 + (1 - T^{-\alpha-\beta}) \frac{(\alpha - \gamma)(\beta - \delta)}{(\alpha + \beta)(\gamma + \delta)}.$$

- The conjecture implies many interesting results about zeros of $\zeta(s)$, such as the pair correlation conjecture of Montgomery.
- By adapting the “recipe” used by Conrey, Farmer, Keating, Rubinstein and Snaith to conjecture asymptotic formulas for moments, one can make the following conjectures.

Conjecture (Conrey, Farmer, Zirnbauer, 2007)

$$\begin{aligned} & \int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)} dt \\ &= \int_0^T \left(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)} A(\alpha, \beta, \gamma, \delta) \right. \\ & \left. + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)} A(-\beta, -\alpha, \gamma, \delta) \right) dt + O\left(T^{1/2+\epsilon}\right), \end{aligned}$$

where

$$A(\alpha, \beta, \gamma, \delta) = \prod_p \frac{\left(1 - \frac{1}{p^{1+\gamma+\delta}}\right) \left(1 - \frac{1}{p^{1+\beta+\gamma}} - \frac{1}{p^{1+\alpha+\delta}} + \frac{1}{p^{1+\gamma+\delta}}\right)}{\left(1 - \frac{1}{p^{1+\beta+\gamma}}\right) \left(1 - \frac{1}{p^{1+\alpha+\delta}}\right)}$$

for $|\Re\alpha|, |\Re\beta| < 1/4$,

$$\frac{1}{\log T} \ll \Re\gamma, \Re\delta < 1/4, \Im\alpha, \Im\beta, \Im\gamma, \Im\delta \ll T^{1-\epsilon}.$$

The Ratios Conjecture for quadratic Dirichlet L -functions

Conjecture (Conrey, Farmer, Zirnbauer, 2007)

$$\sum_{d \leq X}^* \frac{L(1/2 + \alpha, \chi_d)}{L(1/2 + \beta, \chi_d)} = \sum_{d \leq X}^* \left(\frac{\zeta(1 + 2\alpha)}{\zeta(1 + \alpha + \beta)} A(\alpha, \beta) + \left(\frac{d}{\pi}\right)^{-\alpha} \frac{\Gamma(1/4 - \alpha/2)}{\Gamma(1/4 + \alpha/2)} \frac{\zeta(1 - 2\alpha)}{\zeta(1 - \alpha + \beta)} A(-\alpha, \beta) \right) + O\left(X^{1/2+\epsilon}\right),$$

where

$$A(\alpha, \beta) = \prod_p \left(1 - \frac{1}{p^{1+\alpha+\beta}}\right)^{-1} \left(1 - \frac{1}{(p+1)p^{1+2\alpha}} - \frac{1}{(p+1)p^{\alpha+\beta}}\right),$$

for $|\Re\alpha| < 1/4$,

$$\frac{1}{\log X} \ll \Re\beta < 1/4, \quad \Im\beta \ll X^{1-\epsilon}.$$

- M. Cech (2021): restricted range of parameters, nonuniform

The Ratios Conjecture in Random Matrix Theory

One can compute ratios of characteristic polynomials in matrix ensembles:

- Conrey-Farmer-Zirnbauer
- Borodin-Strahov
- Conrey-Forrester-Snaith
- Bump-Gamburd
- Huckleberry-Puttmann-Zirnbauer

Theorem (Conrey-Farmer-Zirnbauer)

For $\Re \gamma_k > 0$, we have

$$\int_{U_{Sp(2N)}} \frac{\prod_{k=1}^K \Lambda_A(e^{-\alpha_k})}{\prod_{k=1}^K \Lambda_A(e^{-\gamma_k})} dA$$
$$= \sum_{\epsilon \in \{-1,1\}^K} e^{N \sum_{k=1}^K (\epsilon_k \alpha_k - \alpha_k)} \frac{\prod_{j \leq k \leq K} z(\epsilon_j \alpha_j + \epsilon_k \alpha_k) \prod_{q < r \leq K} z(\gamma_q + \gamma_r)}{\prod_{k=1}^K \prod_{q=1}^K z(\epsilon_k \alpha_k + \gamma_q)},$$

where $z(x) = (1 - e^{-x})^{-1}$.

Applications of the Ratios Conjecture

- Compute the one-level density of zeros in families of L -functions, for test functions whose Fourier transforms have any support.

Conjecture (Chowla's conjecture)

$L(1/2, \chi) \neq 0$ for any χ a Dirichlet character.

- Soundararajan: $\geq 87.5\%$ of $L(1/2, \chi_d) \neq 0$
- Ozluk-Snyder: $\geq 93.75\%$ of $L(1/2, \chi_d) \neq 0$ by computing the one-level density of zeros with support $(-2, 2)$ (GRH)
- The Ratios Conjecture $\Rightarrow 100\%$ of $L(1/2, \chi_d) \neq 0$
- Compute the lower order terms for the pair correlation of the zeros of $\zeta(s)$, which were previously heuristically computed by Bogomolny and Keating.
- Compute mollified moments of $\zeta(s)$ or other L -functions
- Obtain conjectures for moments of $|\zeta'(\rho)|$
- Etc.

Negative moments of $\zeta(s)$

Conjecture (Gonek, 1989)

Let $k > 0$ be fixed. Uniformly for $\frac{1}{\log T} \leq \delta \leq 1$, we have

$$\int_1^T \left| \zeta\left(\frac{1}{2} + \delta + it\right) \right|^{-2k} dt \asymp T \left(\frac{1}{\delta}\right)^{k^2},$$

and uniformly for $0 < \delta \leq \frac{1}{\log T}$, we have

$$\int_1^T \left| \zeta\left(\frac{1}{2} + \delta + it\right) \right|^{-2k} dt \asymp \begin{cases} T(\log T)^{k^2} & \text{if } k < 1/2 \\ \log(e/(\delta \log T)) T(\log T)^{k^2} & \text{if } k = 1/2 \\ (\delta \log T)^{1-2k} T(\log T)^{k^2} & \text{if } k > 1/2. \end{cases}$$

- Random matrix theory computations (Berry-Keating; Forrester-Keating) suggest transition regimes when $k = (2n + 1)/2$, for n a positive integer

Negative moments of $\zeta(s)$

- Gonek obtained **lower bounds** consistent with the conjecture for all $k > 0$ in the range $\frac{1}{\log T} \leq \delta \leq 1$ and for $k < 1/2$ in the range $0 < \delta \leq \frac{1}{\log T}$.
- **Upper bounds??**
- Work in progress with H. Bui: upper bounds when $\log(1/\delta) \ll \log \log T$.

Dictionary

\mathbb{N}	monic polynomials in $\mathbb{F}_q[t]$
$ n $	$ f := q^{\deg(f)}$
primes p	monic irreducible polynomials P

Define the zeta-function

$$\zeta_q(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{1 - q^{1-s}}.$$

$$u = q^{-s} : \mathcal{Z}(u) = \sum_{f \text{ monic}} u^{\deg(f)} = \frac{1}{1 - qu}.$$

Function fields background

- Let χ be a primitive character (mod h). The L -function associated to χ is defined by

$$L(s, \chi) = \mathcal{L}(u, \chi) = \sum_{f \text{ monic}} \chi(f) u^{\deg(f)} = \prod_{P \nmid h} \left(1 - \chi(P) u^{\deg(P)}\right)^{-1}.$$

- $\mathcal{L}(u, \chi)$ satisfies the following:
 - It is a **polynomial** of degree $\leq \deg(h) - 1$.
 - It has a functional equation. If χ is an odd character, then

$$\mathcal{L}(u, \chi) = \omega(\chi) (\sqrt{qu})^{\deg(h)-1} \mathcal{L}\left(\frac{1}{qu}, \bar{\chi}\right),$$

where $\omega(\chi)$ is the root number.

- All the nontrivial zeros lie on the circle of radius $|u| = \frac{1}{\sqrt{q}}$ (RH).

The Ratios Conjecture over function fields

\mathcal{H}_n = squarefree polynomials of degree n in $\mathbb{F}_q[t]$.

Conjecture (Andrade, Keating, 2014)

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \frac{L(1/2 + \alpha, \chi_D)}{L(1/2 + \beta, \chi_D)} = \frac{\zeta_q(1 + 2\alpha)}{\zeta_q(1 + \alpha + \beta)} A(\alpha, \beta) \\ + q^{-2g\alpha} \frac{\zeta_q(1 - 2\alpha)}{\zeta_q(1 - \alpha + \beta)} A(-\alpha, \beta) + O(q^{-g+\epsilon g}),$$

where

$$A(\alpha, \beta) = \prod_P \left(1 - \frac{1}{|P|^{1+\alpha+\beta}}\right)^{-1} \left(1 - \frac{1}{(|P| + 1)|P|^{1+2\alpha}} - \frac{1}{(|P| + 1)|P|^{\alpha+\beta}}\right),$$

and for

$$|\Re\alpha| < \frac{1}{4}, \quad \frac{1}{g} \ll \Re\beta < \frac{1}{4}.$$

Heuristic arguments

- Using the approximate functional equation:

$$L(1/2 + \alpha, \chi_D) = \sum_{\deg(f) \leq g} \frac{\chi_D(f)}{|f|^{1/2+\alpha}} + q^{-2g\alpha} \sum_{\deg(f) \leq g-1} \frac{\chi_D(f)}{|f|^{1/2-\alpha}}.$$

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$$L(1/2 + \beta, \chi_D)^{-1} = \sum_{h \text{ monic}} \frac{\mu(h)\chi_D(h)}{|h|^{1/2+\beta}}$$

- The first piece from the approximate functional equation gives

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{f,h} \frac{\mu(h)}{|h|^{1/2+\beta}|f|^{1/2+\alpha}} \sum_{D \in \mathcal{H}_{2g+1}} \chi_D(fh),$$

and keep $fh = \square$. Rewrite $f \mapsto f^2h$.

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$$\sum_{f,h} \frac{\mu(h)}{|h|^{1+\alpha+\beta}|f|^{1+2\alpha}} \prod_{P|fh} \left(1 + \frac{1}{|P|}\right)^{-1} \Rightarrow \frac{\zeta_q(1+2\alpha)}{\zeta_q(1+\alpha+\beta)} A(\alpha, \beta).$$

The Ratios Conjecture over function fields

Theorem (Bui, F., Keating, 2021)

- For $\frac{1}{g^{\frac{1}{2}-\epsilon}} \ll \Re\beta$, and $|\Re\alpha| < 1/2$, we have

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \frac{L(1/2 + \alpha, \chi_D)}{L(1/2 + \beta, \chi_D)} = \frac{\zeta_q(1 + 2\alpha)}{\zeta_q(1 + \alpha + \beta)} A(\alpha, \beta) \\ + q^{-2g\alpha} \frac{\zeta_q(1 - 2\alpha)}{\zeta_q(1 - \alpha + \beta)} A(-\alpha, \beta) + O\left(q^{-g\Re\beta(3+2\alpha-\epsilon)}\right).$$

- $k = 2$: RC for $\frac{1}{g^{1/4-\epsilon}} \ll \Re\beta_1, \Re\beta_2$ and $|\Re\alpha_1|, |\Re\alpha_2| < \frac{1}{4}$.
- $k = 3$: RC for $\frac{1}{g^{1/6-\epsilon}} \ll \Re\beta_1, \Re\beta_2, \Re\beta_3$ and $|\Re\alpha_1|, |\Re\alpha_2|, |\Re\alpha_3| < \frac{1}{16}$.

Upper bounds for negative moments of L -functions

The main ingredients in the proof are:

- Obtaining asymptotic formulas for twisted, shifted moments of L -functions
- Obtaining upper bounds for negative moments of L -functions; builds on work of Soundararajan, Harper, Radziwill-Soundararajan

Theorem (Bui, F., Keating, 2021)

Let k be a positive integer such that $k > 1/2$. Let $\frac{1}{g^{2k-\epsilon}} \ll \beta$. Then

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \frac{1}{|L(1/2 + \beta + it, \chi_D)|^k} \ll \left(\frac{1}{\beta}\right)^{k(k-1)/2} (\log g)^{k(k+1)/2}.$$

Theorem (F., 2021)

The bound above and the Ratios Conjecture hold for $\beta \gg \frac{\log g}{g}$.

Ideas of proof

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} \frac{L(1/2 + \alpha, \chi_D)}{L(1/2 + \beta, \chi_D)} = \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} L(1/2 + \alpha, \chi_D) \\ \times \sum_{h \text{ monic}} \frac{\mu(h)}{|h|^{1/2+\beta}} \chi_D(h)$$

For some parameter X to be chosen later, let

$$S_{\leq X} = \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} L(1/2 + \alpha, \chi_D) \sum_{\deg(h) \leq X} \frac{\mu(h)}{|h|^{1/2+\beta}} \chi_D(h),$$

$$S_{> X} = \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} L(1/2 + \alpha, \chi_D) \sum_{\deg(h) > X} \frac{\mu(h)}{|h|^{1/2+\beta}} \chi_D(h).$$

Using Perron's formula for the sum over h , we rewrite

$$S_{>X} = \frac{1}{|\mathcal{H}_{2g+1}|} \frac{1}{2\pi i} \oint \sum_{D \in \mathcal{H}_{2g+1}} \frac{L(1/2 + \alpha, \chi_D)}{\mathcal{L}\left(\frac{z}{q^{1/2+\beta}}, \chi_D\right) z^X (z-1)} \frac{dz}{z},$$

where we are integrating over a circle $|z| > 1$. We pick $|z| = q^{\Re\beta/2}$, use Holder's inequality for the sum over D and then use upper bounds for negative moments of L -functions.

A key inequality

We have

$$\begin{aligned} \log|L(1/2 + \beta + it, \chi_D)| &\geq \frac{2g}{N+1} \log\left(1 - q^{-(N+1)\beta}\right) \\ &+ \Re\left(\sum_{\deg(P) \leq N} \frac{\chi_D(P)}{|P|^{1/2+\beta+it}}\right) + \Re\left(\frac{1}{2} \sum_{\deg(P) \leq N/2} \frac{1}{|P|^{1+2\beta+2it}}\right) + O(1), \end{aligned}$$

so

$$\begin{aligned} \frac{1}{|L(1/2 + \beta + it, \chi_D)|^k} &\leq \sqrt{N}^k \left(\frac{1}{1 - q^{-(N+1)\beta}}\right)^{\frac{2gk}{N+1}} \\ &\times \exp\left(k \sum_{\deg(P) \leq N} \frac{\chi_D(P)a(P)}{|P|^{1/2+\beta}}\right), \end{aligned}$$

where $a(P) = -\cos(t \deg(P) \log q)$.

Think of $N \approx \frac{\log g}{\beta}$.

The upper bound

- Pointwise bound

$$\frac{1}{|L(1/2 + \beta + it, \chi_D)|} \leq \left(\frac{1}{1 - g^{-2\beta}} \right)^{\frac{(1+\epsilon)g}{\log_q g}}.$$

- Use the inequality

$$e^t \leq (1 + e^{-\ell/2}) \sum_{s \leq \ell} \frac{t^s}{s!},$$

for $t \leq \ell/e^2$ and ℓ even.

- Split the primes into K intervals, where $K \sim c \log(1/\beta)$:

$$I_0 = (0, N_0], I_1 = (N_0, N_1], \dots, I_K = (N_{K-1}, N_K].$$

The upper bound

- If the contribution from primes in l_0 is “big” (call this set of discriminants \mathcal{T}_0) i.e: if

$$\left| \sum_{\deg(P) \in l_0} \frac{\chi_D(P)a(P)}{|P|^{1/2+\beta}} \right| > \frac{l_0}{ke^2},$$

we want to exploit the fact that $|\mathcal{T}_0|$ is small.

$$\begin{aligned} \sum_{D \in \mathcal{T}_0} \frac{1}{|L(1/2 + \beta + it, \chi_D)|^k} &\leq \sum_{D \in \mathcal{H}_{2g+1}} \frac{1}{|L(1/2 + \beta + it, \chi_D)|^k} \\ &\times \left(\frac{ke^2}{l_0} \sum_{\deg(P) \in l_0} \frac{\chi_D(P)a(P)}{|P|^{1/2+\beta}} \right)^{s_0}, \end{aligned}$$

for some even parameter s_0 for which $s_0 N_0 \leq g$.

The upper bound



$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1}} \frac{1}{|L(1/2 + \beta + it, \chi_D)|^k} \left(\frac{ke^2}{\ell_0} \sum_{\deg(P) \in l_0} \frac{\chi_D(P)a(P)}{|P|^{1/2+\beta}} \right)^{s_0} \\ & \leq \left(\sum_{D \in \mathcal{H}_{2g+1}} \frac{1}{|L(1/2 + \beta + it, \chi_D)|^{2k}} \right)^{1/2} \frac{k^{s_0} e^{2s_0}}{\ell_0^{s_0}} \\ & \times \left(\sum_{D \in \mathcal{H}_{2g+1}} \left(\sum_{\deg(P) \in l_0} \frac{\chi_D(P)a(P)}{|P|^{1/2+\beta}} \right)^{2s_0} \right)^{1/2} \end{aligned}$$

- For the first term, use the pointwise bound for the L -function; for the second term, we compute the moments and keep the diagonal pieces. Choose $\ell_0 = s_0$.

The upper bound



$$\begin{aligned} & \sum_{D \in \mathcal{M}_{2g+1}} \left(\sum_{\deg(P) \in l_0} \frac{\chi_D(P) a(P)}{|P|^{1/2+\beta}} \right)^{2s_0} \\ &= (2s_0)! \sum_{D \in \mathcal{M}_{2g+1}} \sum_{\substack{\Omega(f)=2s_0 \\ P|f \Rightarrow \deg(P) \in l_0}} \frac{\chi_D(f) a(f) \nu(f)}{|f|^{1/2+\beta}}, \end{aligned}$$

where ν is the multiplicative function given by $\nu(P^a) = 1/a!$.

- Only keep $f = \square$. Get

$$\ll q^{2g} \frac{(2s_0)!}{s_0!} \left(\sum_{\deg(P) \in l_0} \frac{1}{|P|} \right)^{s_0} \ll q^{2g} \frac{(2s_0)!}{s_0!} (\log N_0)^{s_0}.$$

The upper bound



$$\sum_{D \in T_0} \frac{1}{|L(1/2 + \beta + it, \chi_D)|^k} \ll q^{2g} \left(\frac{1}{1 - g^{-2\beta}} \right)^{\frac{(1+\epsilon)g}{\log q g}} \\ \times \exp\left(-\frac{s_0}{2} \log s_0\right) \exp(cs_0 \log \log N_0) = o(q^{2g}),$$

for

$$s_0 = 2 \left[\frac{(2 + 3\epsilon)g \log q}{2(\log g)^2} \log \left(\frac{1}{1 - g^{-2\beta}} \right) \right], \quad N_0 = [g/s_0], \\ N_j = rN_{j-1}, \quad s_j = [ag/N_j].$$

The upper bound

- If the contribution from primes in I_0 is “small”, move to the next interval and proceed as before.
- ...
- The contribution from the term for which the sums over each interval are small will be bounded by

$$\begin{aligned} &\ll \exp\left(\frac{2gk}{N_K} \log\left(\frac{1}{1 - q^{-N_K\beta}}\right)\right) N_K^{\frac{k}{2}} \prod_{r=0}^K \sum_{s \leq \ell_r} \frac{1}{s!} \left(\sum_{\deg(P) \in I_r} \frac{k \chi_D(P) a(P)}{|P|^{1/2+\beta}} \right)^s \\ &\ll \exp\left(\frac{2gk}{N_K} \log\left(\frac{1}{1 - q^{-N_K\beta}}\right)\right) N_K^{\frac{k}{2}} \\ &\quad \times \prod_{r=0}^K \prod_{\substack{P | f_r \Rightarrow \deg(P) \in I_r \\ \Omega(f_r) \leq \ell_r}} \frac{k^{\Omega(f_r)} \chi_D(f_r) a(f_r) \nu(f_r)}{|f_r|^{1/2+\beta}}, \end{aligned}$$

where ν is the multiplicative function defined by $\nu(P^a) = 1/a!$.

The upper bound

- Keeping the $f_r = \square$, this will be

$$\begin{aligned} &\ll \exp\left(\frac{2gk}{N_K} \log\left(\frac{1}{1 - q^{-N_K\beta}}\right)\right) N_K^{k/2} \prod_{\deg(P) \leq N_K} \left(1 - \frac{k^2}{2|P|}\right)^{-1} \\ &\ll \exp\left(\frac{2gk}{N_K} \log\left(\frac{1}{1 - q^{-N_K\beta}}\right)\right) N_K^{k^2/2 + k/2}. \end{aligned}$$

- With the choice $N_K \approx \frac{\log g}{\beta}$, the bound follows.

The asymptotic formula for $S_{\leq X}$

Recall

$$S_{\leq X} = \frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} L(1/2 + \alpha, \chi_D) \sum_{\deg(h) \leq X} \frac{\mu(h)}{|h|^{1/2+\beta}} \chi_D(h),$$

so we need asymptotic formulas for

$$\sum_{D \in \mathcal{H}_{2g+1}} L(1/2 + \alpha, \chi_D) \chi_D(h)$$

for h of small enough degree.

- $\alpha = 0, h = 1$: Andrade-Keating; F.
- $\alpha = 0$: Bui-F.

Sketch of proof

Use standard techniques such as

- Approximate functional equation
- Poisson summation
- Upper bounds for positive moments of L -functions, etc.
- Approximate functional equation

$$L\left(\frac{1}{2}, \chi_D\right) = \sum_{f \in \mathcal{M}_{\leq g}} \frac{\chi_D(f)}{\sqrt{|f|}} + \sum_{f \in \mathcal{M}_{\leq g-1}} \frac{\chi_D(f)}{\sqrt{|f|}}.$$

- $$\sum_{D \in \mathcal{H}_{2g+1}} \chi_D(f) = \sum_{C|f^\infty} \sum_{h \in \mathcal{M}_{2g+1-2d(C)}} \chi_f(h) - q \sum_{C|f^\infty} \sum_{h \in \mathcal{M}_{2g-1-2d(C)}} \chi_f(h)$$

Poisson summation formula

- For $a \in \mathbb{F}_q((\frac{1}{x}))$, $a = \sum_{i=-\infty}^{\infty} a_i \left(\frac{1}{x}\right)^i$, define

$$e(a) = e^{\frac{2\pi i a_1}{q}}.$$

- Generalized Gauss sums $G(V, \chi_f) = \sum_{u \pmod{f}} \chi_f(u) e\left(\frac{uV}{f}\right)$.
- If $\deg(f) = n$ is even, then

$$\sum_{h \in \mathcal{M}_m} \chi_f(h) = \frac{q^m}{|f|} \left[G(0, \chi_f) + (q-1) \sum_{V \in \mathcal{M}_{\leq n-m-2}} G(V, \chi_f) - \sum_{V \in \mathcal{M}_{n-m-1}} G(V, \chi_f) \right]$$

Setup

$$\sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right) = S_g + S_{g-1} + O(q^{\frac{g}{2}(1+\epsilon)}),$$

where

$$S_g = q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq g} \\ \deg(f) \text{ even}}} \frac{1}{|f|} \sum_{C|f^\infty} \frac{1}{|C|^2} \left[G(0, \chi_f) + \right. \\ \left. (q-1) \sum_{V \in \mathcal{M}_{\leq \deg(f)-2g-3+2d(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}} - \sum_{V \in \mathcal{M}_{\deg(f)-2g-2+2d(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}} \right]$$

- $G(0, \chi_f) \neq 0$ iff $f = \square$, in which case $G(0, \chi_f) = \phi(f)$.

Main term from $V = 0$

$$\begin{aligned}M_{g-1} &= q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq g-1} \\ \deg(f) \text{ even}}} \frac{1}{|f|} \sum_{C|f^\infty} \frac{1}{|C|^2} G(0, \chi_f) \\ &= \frac{q^{2g+1}}{\zeta_q(2)} \sum_{l \in \mathcal{M}_{\leq \lfloor \frac{g-1}{2} \rfloor}} \frac{1}{|l|} \left(\prod_{P|l} \frac{|P|}{|P|+1} \right) \\ &= \frac{q^{2g+1}}{\zeta_q(2)} \frac{1}{2\pi i} \oint_{|u| < \frac{1}{q}} C(u) \frac{(qu)^{-\lfloor \frac{g-1}{2} \rfloor} du}{(1-qu)^2 u}, \\ C(u) &= \prod_P \left(1 - \frac{u^{d(P)}}{|P|+1} \right).\end{aligned}$$

$$V \neq 0$$

$$\sum_f \frac{G(V, \chi_f)}{\sqrt{|f|}} w^{d(f)} = \mathcal{L}(w, \chi_V) \prod_P \mathcal{R}_P(w).$$

\Rightarrow Focus on $V = \square = l^2$.

$$S_g(V = \square) = q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq g} \\ \deg(f) \text{ even}}} \frac{1}{|f|} \sum_{C|f^\infty} \frac{1}{|C|^2} \left[(q-1) \cdot \right.$$

$$\left. \sum_{l \in \mathcal{M}_{\leq \frac{\deg(f)}{2} - g - 2 + d(C)}} \frac{G(l^2, \chi_f)}{\sqrt{|f|}} - \sum_{l \in \mathcal{M}_{\frac{\deg(f)}{2} - g - 1 + d(C)}} \frac{G(l^2, \chi_f)}{\sqrt{|f|}} \right]$$

- For the term $S_g(V = \square)$, since $C|f^\infty$, we roughly have $\deg(C) \leq \deg(f)$. Combining this with the condition on the degree of l implies that $\deg(f) \geq 2g/3$. Thus, heuristically we have a sum of the form $\sum_{\frac{2g}{3} \leq \deg(f) \leq g} 1$.

$$M_{g-1} = \frac{q^{2g+1}}{\zeta_q(2)} \oint_{|u| < \frac{1}{q}} C(u) \frac{(qu)^{-\lfloor \frac{g-1}{2} \rfloor}}{(1-qu)^2} \frac{du}{u} \quad \left| \quad S_g(V = \square) = q^{\frac{2g+1}{3}} P_1(g) \right.$$

$$\left. - \frac{q^{2g+1}}{\zeta_q(2)} \oint_{|u| = \sqrt{q}} C(u) \frac{(qu)^{-\lfloor \frac{g-1}{2} \rfloor}}{(1-qu)^2} \frac{du}{u} \right.$$

$$M_{g-1} + S_g(V = \square) = -\text{Res}\left(u = \frac{1}{q}\right) + q^{\frac{2g+1}{3}} P_1(g).$$

- By matching up the terms, we get rid of the error of size $q^{g(1+\epsilon)}$.
- The term corresponding to $V \neq \square$ is bounded by $q^{\frac{g}{2}(1+\epsilon)}$.

Thank you!