Worldsheet correlators in AdS_3 & Hurwitz theory

Lorenz Eberhardt

IAS Princeton

based on work with Andrea Dei

Motivation

- String theory on AdS is interesting & important for holography

- For AdS_3 & pure NS-NS flux, string theory can be described using an $SL(2,\mathbb{R})$ WZW model [Giveon, Kutasov, Seiberg '98; Maldacena, Ooguri '00; ...]

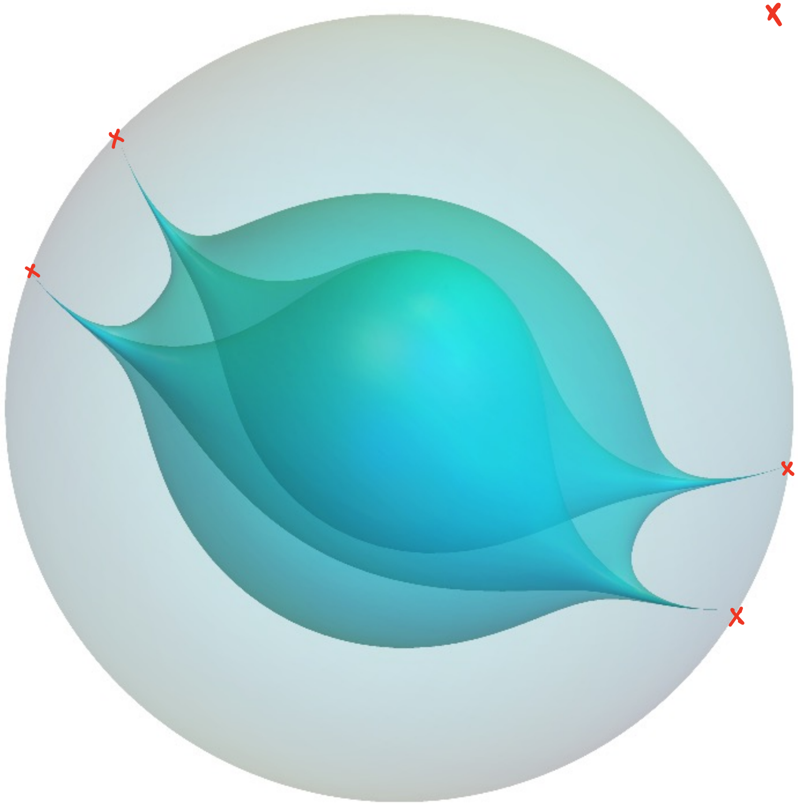
- There seems to be a general feeling that the $SL(2,\mathbb{R})$ WZW model should be solvable, but so far only partially achieved....

- In this talk we take a step towards the full solution of string theory on (E)AdS_3 with pure NS-NS flux.

- Intriguing mathematical structure!

What's the difficulty?

- We are interested in holographic global AdS correlators:



x : vertex operator insertions

- The string can have non-trivial winding (= spectral flow) around the insertion points
- They are difficult to compute from the worldsheet
- The winding correlators contain most of the interesting physics!

The worldsheet CFT

- Restrict to bosonic strings

- The symmetry algebra of the Euclidean AdS_3 worldsheet theory is

$$[J^3_m, J^3_n] = -\frac{k}{2} m \, \delta_{m+n,0}$$

$$[J^3_m, J^\pm_n] = \pm J^\pm_{m+n}$$

$$[J^+_m, J^-_n] = k m \, \delta_{m+n,0} - 2 J^3_{m+n}.$$

$$[\alpha^\mu_m, \alpha^\nu_n] = m \eta^{\mu\nu} \delta_{m+n,0}$$

$$(\times 2)$$

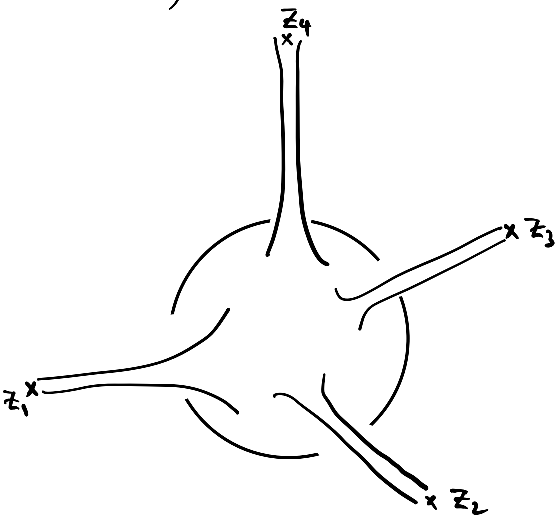
$$J^+_{-2} J^3_{-1} | \ldots \rangle$$

$\Rightarrow$ Everything will be a consequence of this algebra!

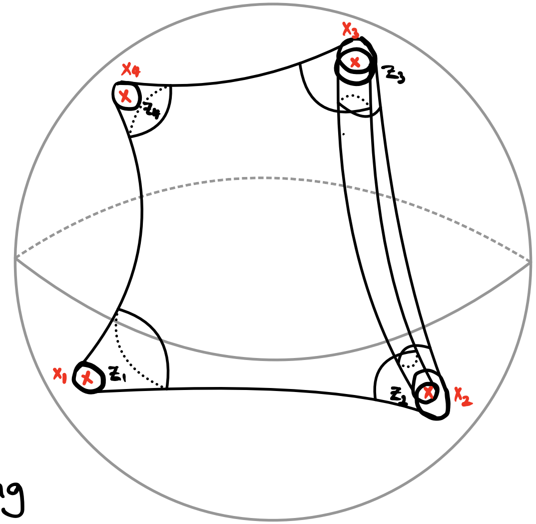
Vertex operators

- What are the vertex operators of which we want to compute correlators?

flat space



EAdS$_3$



$$V(p,z) = e^{ipX(z)}$$

external momentum

worldsheet position

$$\prod_{i} |z_i - z_j|^{-2p_i \cdot p_j}$$

$$\Delta = \frac{\alpha'}{4} p^2$$

winding

$$V^w_{h,\Delta}(x,z)$$

worldsheet position

spacetime position

spacetime

worldsheet conformal weight

$$\sim V^w_{j,h}(x,z) \qquad \Delta = -\frac{j(j-1)}{k-2} - wh + \frac{kw^2}{4}.$$

$$j \in \mathbb{R} \cup \left(\tfrac{1}{2} + i\mathbb{R}\right)$$

Vertex operators

- The vertex operators $V^w_{j,h,\bar{h}}(x,z)$ are spectrally flowed: not highest weight representations of $sl(2,\mathbb{R})_k$.
- This is why computing correlators is hard
- The unflowed sector (all $w=0$) was solved by Teschner. [Teschner '98-'00]
- Partial results about flowed sector are known
  [Maldacena, Ooguri '01; Giribet, Nuñez '00; ...]
- We are going to trade $(h,\bar{h})$ for a third position variable $y \in \mathbb{CP}^1$.
  This is just an integral transform.

$$V^w_{j,h,\bar{h}}(x,z) = \int \frac{d^2y}{(2\pi)^2}\, y^{\frac{kw}{2}+j-h-1}\, \bar{y}^{\frac{kw}{2}+j-\bar{h}-1}\, V^w_j(x,y,z)$$

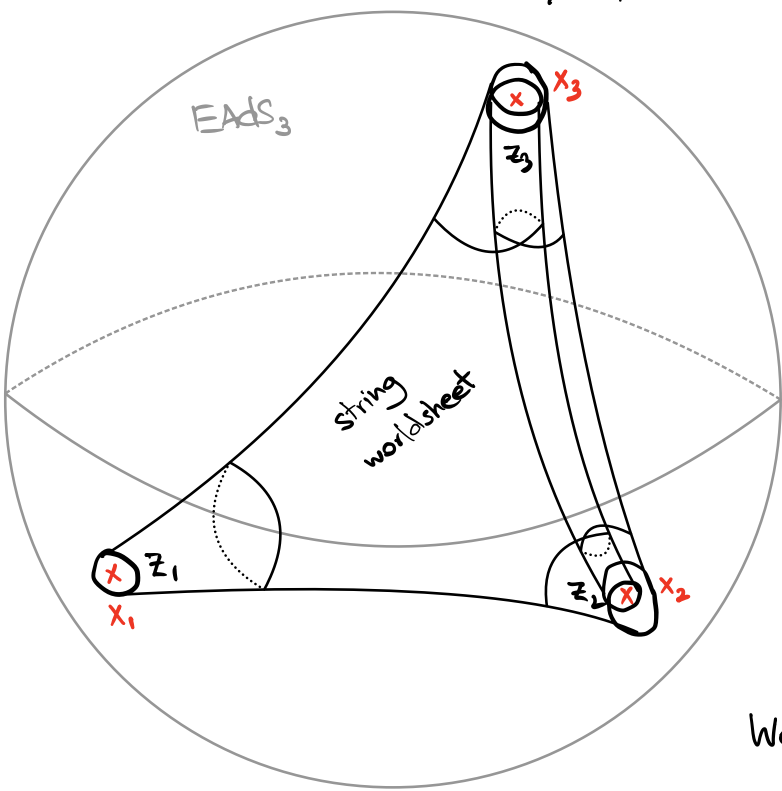
$\Rightarrow$ Consider $V^w_j(x,y,z)$.

Correlators

- We want to compute correlators:

$$\left\langle \prod_{i=1}^{n} V_{j_i}^{w_i}(x_i,\bar{x}_i,y_i,\bar{y}_i,z_i,\bar{z}_i)\right\rangle$$ on the Riemann sphere

- Main trouble: so many parameters... We mainly use symmetries to constrain the answer



$$= \left\langle V_{j_1}^{1}(x_1,y_1,z_1)\, V_{j_2}^{2}(x_2,y_2,z_2)\, V_{j_3}^{2}(x_3,y_3,z_3)\right\rangle.$$

For $n\geq 4$, we also need to integrate over moduli

We restrict to $n\leq 4$.

Ward identities

- Global Ward identities allow us to set

$$x_1 = z_1 = 0 \quad , \qquad x_2 = z_2 = 1 \,, \qquad x_3 = z_3 = \infty .$$

$\Rightarrow$ 3pt function fully fixed, 4pt function depends on two crossratios

$$x = x_4 \,, \quad z = z_4 .$$

- But we have affine symmetry
  - $\Rightarrow$ There are also local Ward identities that restrict the answer further
- They are very complicated and we only know how to derive them for fixed choices of w_i.
- More Ward identities: KZ-equation, null-vector decoupling for special choices of spins j_i.

Branched coverings

- The solution of these constraints encode branched coverings.

Def: A branched cover $\gamma : S^2 \longrightarrow S^2$ is a holomorphic map with finitely many ramification points z_i

$$\gamma(\zeta) = x_i + a_i (\zeta - z_i)^{w_i} + \mathcal{O}((\zeta - z_i)^{w_i+1})$$

w_i : ramification indices.

- For generic choices of $\{x_i, z_i, w_i\}_{i=1\dots n}$, covering maps don't exist.
- Existence puts $n-3$ constraints on $\{x_i, z_i\}_{i=1\dots n}$.
- Riemann-Hurwitz formula:

$$\underset{\chi(S^2)}{\underset{\shortparallel}{2}} = \underset{\chi(S^2)}{\underset{\shortparallel}{2}} \deg \gamma - \sum_i (w_i - 1)$$

Example :

- Let's find the covering map for $w_1 = 2, w_2 = 2, w_3 = w_4 = 1.$

  $x_1 = z_1 = 0$, $x_2 = z_2 = 1$, $x_3 = z_3 = \infty$, $x_4 = X$, $z_4 = Z.$

- Such a map is necessarily of the form

$$\gamma(\zeta) = \frac{b_1 \zeta^2 + b_2 \zeta + b_3}{c_1 \zeta^2 + c_2 \zeta + c_3} \qquad \gamma'(\zeta) = 0.$$

  because $\deg \gamma = 2$ by the Riemann-Hurwitz formula.

- Impose Taylor expansion at z_1, z_2, z_3:

$$\gamma(\zeta) = \frac{\zeta^2}{2\zeta - 1}$$

- Taylor expansion at z_4 :

$$\gamma(z) = \frac{z^2}{2z - 1} \overset{!}{=} X$$

$\Rightarrow$ covering map existence gives

$$z^2 - 2zX + X = 0 \quad \subset \mathbb{CP}^1 \times \mathbb{CP}^1$$

Polynomial $P_w(x,z)$

- For $n=4$, this constraint is encoded in a single 'polynomial' in (x,z) e.g.

$$P_{(2,2,1,1)}(x,z) = \underbrace{z^{-1}\,(1-z)^{-1}}_{\text{useful normalization}}\,(z^2-2zx+x)$$

useful normalization, plays no role at this point since $x,z \neq 0,1,\infty$

$$P_{(2,2,3,3)}(x,z) = z^{-3}(1-z)^{-3}\,(z^6-6xz^5+15xz^4-20xz^3+6x^2z^2+9xz^2$$
$$-6x^2z+x^2)$$

$\vdots$

- There is an efficient algorithm to compute these   See also [Pakman, Rastelli, Razamat '09]
- P_w is only defined for $\sum_{i=1}^{4} w_i$ even
- They satisfy many magic identities that we experimentally observe!
- Not studied in math literature ...

Solution of the local Ward identities

- A solution of the local Ward identities only exists for

$$\sum_{i\neq j}(w_i+1) \geq w_j+1 \quad \forall j.$$

$\Rightarrow$ winding is violated by at most one unit [Maldacena, Ooguri '01]

- Define for $I \subset \{1,2,3,4\}$

$$X_I = (\text{some simple prefactor}) \times \sum_{i\in I:\, \varepsilon_i=\pm 1} \mathcal{P}_{w+\sum_{i\in I}\varepsilon_i e_i} \prod_{i\in I} y_i^{\frac{1-\varepsilon_i}{2}}.$$

e_i : unit vector

e.g.

$$X_\emptyset = \mathcal{P}_w$$

$$X_{12} = \mathcal{P}_{(w_1+1,w_2+1,w_3,w_4)} + y_1 \mathcal{P}_{(w_1-1,w_2+1,w_3,w_4)} + y_2 \mathcal{P}_{(w_1+1,w_2-1,w_3,w_4)} + y_1 y_2 \mathcal{P}_{(w_1-1,w_2-1,w_3,w_4)}$$

$$X_3 = \mathcal{P}_{(w_1,w_2,w_3+1,w_4)} + y_3 \mathcal{P}_{(w_1,w_2,w_3-1,w_4)}$$

Solution of the local Ward identities

- General solution takes easy form in terms of X_I:

$\sum_{i=1}^{4} w_i$ even (similar formula for odd)

$$\left\langle \prod_{i=1}^{4} V_{j_i}^{w_i}(x_i, y_i, z_i) \right\rangle = X_{\emptyset}^{j_1+j_2+j_3+j_4-k} \; X_{12}^{-j_1-j_2+j_3-j_4} \; X_{13}^{-j_1+j_2-j_3+j_4}$$
$$\times X_{23}^{+j_1-j_2-j_3+j_4} \; X_{34}^{-2j_4} \; F\left(\frac{X_{23}X_{14}}{X_{12}X_{34}}, z \right)$$

checked in many cases, but no idea how to prove!

- Same as solution to global Ward identities with

$$x_i - x_j \longrightarrow X_{ij}.$$

- $F(c,z)$ is an unknown function that depends on the "generalized crossratio"

$$c = \frac{X_{23}X_{14}}{X_{12}X_{34}}$$

$$1 - c = \frac{X_{24}X_{13}}{X_{12}X_{34}}, \quad c - z = \frac{X_{\emptyset}X_{1234}}{X_{12}X_{34}}$$

highly nontrivial!

Solution of the local Ward identities

- The problem is now "as hard as" the unflowed sector
- Everything is determined in terms of the geometry of branched covers.
- 3pt function is a special case of this ($w_4=0$, $j_4=0$):

$$\left\langle \prod_{i=1}^{3} V_{j_i}^{w_i}(x_i,y_i,z_i)\right\rangle = \begin{cases} C\, X_1^{-j_1+j_2+j_3-\frac{k}{2}}\, X_2^{j_1-j_2+j_3-\frac{k}{2}}\, X_3^{j_1+j_2-j_3-\frac{k}{2}}\, X_{123}^{\frac{k}{2}-j_1-j_2-j_3}, & \sum_{i=1}^{3} w_i \text{ odd} \\ C\, X_{12}^{-j_1-j_2+j_3}\, X_{13}^{-j_1+j_2-j_3}\, X_{23}^{j_1-j_2-j_3}, & \sum_{i=1}^{3} w_i \text{ even} \end{cases}$$

$C \equiv C(j_1,j_2,j_3,w_1,w_2,w_3,k)$ not fixed by symmetry

- In some case we can explicitly transform this back to the h-variables.
- Solution given in terms of an F_A Lauricella hypergeometric function

More constraints!

- We can actually do better!
- Focus on four-point parity even case: $\sum_{i=1}^{4} w_i$ even.
- $F(c,z)$ obeys a further constraint: The Knizhnik-Zamolodchikov equation.
- The KZ-equation for $F(c,z)$ coincides with the KZ-equation for the non-winding correlator!
- This is also true for other possible constraints such as null-vector constraints (checked only for a few examples)
- This motivates the conjecture

$$F(c,z) = C \left\langle V^{0}_{j_1}(0,0)\, V^{0}_{j_2}(1,1)\, V^{0}_{j_3}(\infty,\infty)\, V^{0}_{j_4}(c,z) \right\rangle$$

$$C = C(j_1, j_2, j_3, j_4, w_1, w_2, w_3, w_4, k).$$

A conjecture for the full correlator

- We believe that $C \equiv 1$ from other consistency conditions.
- Main conjecture: $x_1 = z_1 = 0$ $\quad x_2 = z_2 = 1$ $\quad x_3 = z_3 = \infty$

$$x_4 = x \quad , \quad z_4 = z$$

$\sum_{i=1}^{4} w_i$ even:

$$\left\langle \prod_{i=1}^{4} V^{w_i}_{j_i, h_i, \bar{h}_i}(x_i, z_i) \right\rangle = \int \prod_{i=1}^{4} \frac{d^2 y_i}{(2\pi)^2}\, y_i^{\frac{k w_i}{2} - h_i + j_i - 1}\, \bar{y}_i^{\frac{k w_i}{2} - \bar{h}_i + j_i - 1} \left| X_\phi^{j_1+j_2+j_3+j_4-k} \quad X_{12}^{-j_1-j_2+j_3-j_4} \right.$$

$$\left. \times X_{13}^{-j_1+j_2-j_3+j_4} X_{23}^{+j_1-j_2-j_3+j_4} X_{34}^{-2j_4} \right|^2 \left\langle V^0_{j_1}(0,0)\, V^0_{j_2}(1,1)\, V^0_{j_3}(\infty,\infty)\, V^0_{j_4}\!\left(\frac{X_{23} X_{14}}{X_{12} X_{34}}, z \right) \right\rangle$$

- Unflowed correlators give flowed ones for free!
- Fortunately, unflowed correlators are "understood" [Teschner '97-'99]

$\Rightarrow$ This gives a full solution of the model for these correlators!

Properties

- The correlators have unusual properties.
- They have more singularities than the trivial ones when

  $x=0, \quad x=1, \quad x=\infty, \quad z=0, \quad z=1, \quad z=\infty.$

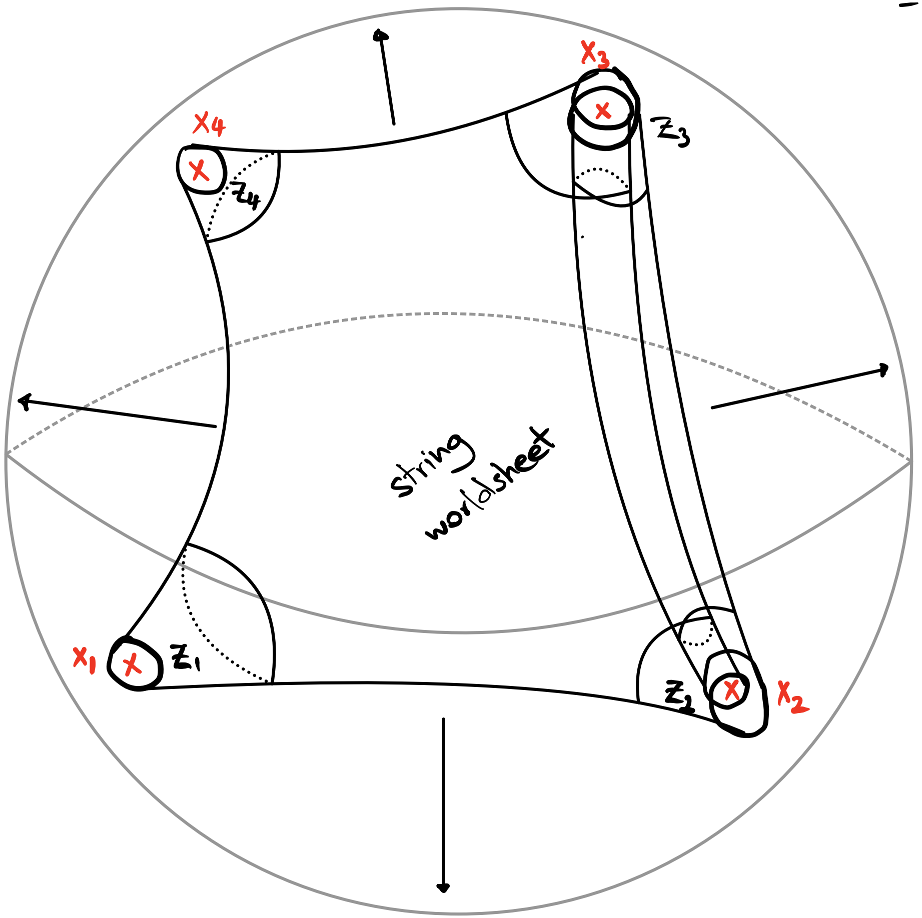
  which correspond to collisions of vertex operators.
- There are singularities occuring in the middle of moduli space for $P_w(x,z)=0$.
- The correlator simplifies close to the singularity $P_w(x, z_*)=0$

$$\left\langle \prod_{i=1}^{4} V^{w_i}_{j_i,h_i,\bar{h}_i}(x_i,z_i) \right\rangle = (\text{computable stuff}) \times |z-z_*|^{2(j_1+j_2+j_3+j_4-k)} + \dots$$

+ more regular

$$\frac{1}{\sum j_i - k + 1}$$

Why is this happening?



- For $P_w(x,z)=0$ the worldsheet can cover the boundary of $EAdS_3$ holomorphically and move outwards

$\Rightarrow$ This leads to a divergence in the worldsheet path integral

[Maldacena, Ooguri '01]

String correlator

- The string moduli space integral is dominated by the singularity if the exponent is tuned.
- Special case: $j_1+j_2+j_3+j_4=5-k$, in which case there is a residue in the string correlator:

normalization of string path integral

$$\operatorname*{Res}_{\sum j_i=5-k}\left\langle\!\left\langle \prod_{i=1}^{4} V^{w_i}_{j_i,h_i,\bar{h}_i}(x_i)\right\rangle\!\right\rangle = C_{S^2}\int d^2z \left\langle \prod_{i=1}^{4} V^{w_i}_{j_i,h_i,\bar{h}_i}(x_i;z_i)\, V^{(i)}_{\text{int}}(z_i)\right\rangle$$

$$= C_{S^2}\sum_{z_*}|z_*|^{-\frac{(4-k-2j_1)(4-k-2j_4)}{2(k-2)}}\,|1-z_*|^{-\frac{(4-k-2j_1)(4-k-2j_4)}{2(k-2)}}\prod_{i=1}^{4} w_i^{\,j_i-\frac{k(w_i+1)}{4}}$$

$$\times \prod_{i=1}^{4} \underbrace{a_i^{\frac{w_i-1}{4}k-h_i}}_{\text{Taylor coefficients of covering map}}\,\bar{a}_i^{\frac{w_i-1}{4}k-\bar{h}_i}\prod_a \underbrace{|\Pi_a|^{-k}}_{\text{Residues of covering map}}\left\langle \prod_{i=1}^{4} V^{(i)}_{\text{int}}(z_{i,*})\right\rangle$$

$$\gamma(z) = x_i + a_i(z-z_i)^{w_i} + \dots$$

More magic identities for P_w needed for this computation...

The dual CFT

- This agrees with the large N limit of the correlator of the symmetric product orbifold [Lunin, Mathur '00; Pakman, Rastelli, Razamat '09; ...]

$$\mathrm{Sym}^N\left(\left[\text{Free boson with background charge } Q = \frac{k-3}{\sqrt{k-2}}\right] \times X\right)$$

$$c = 1 + 6Q^2 = 1 + \frac{6(k-3)^2}{k-2}$$

X : internal CFT of $AdS_3 \times X$

- Momentum of the boson is

$$\alpha_i = \frac{2j_i + k - 4}{2\sqrt{k-2}}$$

$\Rightarrow$ free part of correlator is $\quad z_*^{-2\alpha_1\alpha_4}(1-z_*)^{-2\alpha_2\alpha_4}$

- The condition $j_1 + j_2 + j_3 + j_4 = 5 - k$ is momentum conservation

$$\sum_{i=1}^{4} \alpha_i = \frac{k-3}{\sqrt{k-2}} = Q.$$

The dual CFT

- Our computation gives a direct proof that the singularity is described by the CFT [Seiberg Witten '99; Argurio, Giveon, Shomer '00; LE, Gaberdiel '19].

$$\mathrm{Sym}^N(\mathbb{R}_{Q=\frac{k-3}{\sqrt{k-2}}} \times X).$$

- Advertisement: We believe to know the full (perturbative) spacetime CFT, thus completing the holographic pair.

WIP, stay tuned!

The tensionless limit

- There is an interesting limit one can take in which the description in terms of Sym^N becomes exact. [Dei, LE, Gaberdiel, Gopakumar, Knighton '18–'21, ...]
- This is the superstring on $AdS_3 \times S^3 \times T^4$ for $k=1$ (which corresponds to $k=3$ in the bosonic model). $(k=1) \Leftrightarrow$ small tension
- In this case there are even more constraints.
- Only $j=\frac{1}{2}$ vertex operators are allowed.

$$\Rightarrow \sum_{i=1}^{4} j_i = 2 = 5-k \quad \text{always true}$$

- But the nature of the singularity changes...
- Solution for $j_i = \frac{1}{2}$ & $k=3$:

$$\left\langle \prod_{i=1}^{4} V_{j_i}^{w_i}(x_i, y_i, z_i) \right\rangle = X_\emptyset^{-1} X_{12}^{-1} X_{34}^{-1}$$
$$\times \left\langle V_{\frac{1}{2}}^{0}(0,0)\, V_{\frac{1}{2}}^{0}(1,1)\, V_{\frac{1}{2}}^{0}(\infty,\infty)\, V_{\frac{1}{2}}^{0}\!\left(\frac{X_{23}X_{14}}{X_{12}X_{34}}, z\right) \right\rangle$$

The tensionless limit

- We can replace

$$X_\phi^{-1} \longrightarrow \delta(X_\phi)$$

in the solution & still obtain a solution (because $x\left(\frac{1}{x}\right)' = -\left(\frac{1}{x}\right)$, $x\delta'(x) = -\delta(x)$)

- This is the only solution in this case compatible with all the constraints.

- $\int_{\mathcal{M}_{0,4}}$ integral now receives only contributions from $X_\phi = P_w(x,z) = 0$

$\Rightarrow$ symmetric orbifold answer always exactly true!

- Including fermions etc. one simply gets the large N limit of correlators of $\mathrm{Sym}^N(T^4)$.

Summary

- Full (conjectural) solution of the H_3^+ model with spectral flow (at the moment sphere ≤ 4pt function, but lesson should generalize)

$$\text{unflowed correlator} \xRightarrow[\text{coverings}]{\text{branched}} \text{flowed correlators}$$

- Worldsheet correlators have interesting singularity structure
- Near the singularity, the spacetime correlator becomes the correlator of $\mathrm{Sym}^N(\mathbb{R}_{Q=\frac{k-3}{\sqrt{k-2}}} \times X)$.
- In the tensionless limit, worldsheet correlator has δ-function support & the relation with the symmetric orbifold becomes exact