Divisibility questions on the partition function and their connection to modular forms

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The partition function and Ramanujan congruences

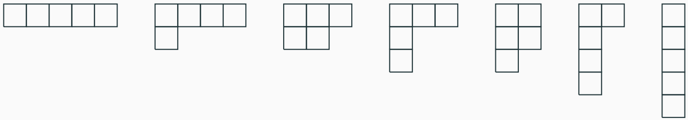
Partition function

Given $n \in \mathbb{R}$, the partition function $p(n)$ records the number of nonincreasing sequences of nonnegative integers that sum up to n.

There are seven partitions of 5, hence $p(5) = 7$:

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

Partitions are sometimes visualized by Young diagrams:



Values of the partition function

The very first values of the partition function can be determined by enumeration, but $p(n)$ growths fast:

$$p(0) = 1, p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(10) = 42,$$
$$p(100) = 190569292, p(200) = 3972999029388.$$

It was an achievement by MacMahon in 1916 to compute $p(n)$ for $n \leq 200$.

Euler's generating function

Partitions are such natural and classical objects that also Euler studied them. He found that

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \quad q \in \mathbb{C}, |q| < 1.$$

To fit notation of modular forms, we will write

$$q = e(\tau) := \exp(2\pi i \tau), \tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0.$$

Then Euler's formula reads

$$\sum_{n=0}^{\infty} p(n)e(n\tau) = \prod_{n=1}^{\infty} \frac{1}{1-e(n\tau)}.$$

Calculating the partition function

Euler's formula yields a recursion for the partition function when combined with Euler's pentagon identity:

$$\prod_{n=1}^{\infty} \big(1 - e(n\tau)\big) = \sum_{n=-\infty}^{\infty} (-1)^n e\Big(\frac{n(3n-1)}{2}\tau\Big).$$

We get

$$\Bigg(\sum_{m \in 6\mathbb{Z}_{\geq 0} + \{0,5\}} - \sum_{m \in 6\mathbb{Z}_{\geq 0} + \{2,3\}}\Bigg) p\big(n - \tfrac{1}{6} m(m+1)\big) = 0.$$

MacMahon based his work on this formula. Nowadays, using the computer algebra system Nemo, we compute $p(200)$ in a naive way without difficulty.

Ramanujan congruences

Ramanujan conjectures in 1919 that for $\ell \in \{5,7,11\}$ and positive integers m

$$p(n) \equiv 0 \pmod{\ell^m}, \text{ i.e., } \ell^m \mid p(n), \quad \text{if } 24n \equiv 1 \pmod{\ell^m}.$$

For $\ell = 11$ and $m = 2$, the first n with $24n \equiv 1$ is 116, so there was exactly one known value of $p(n)$ supporting the conjecture in this case.

The cases $m = 1$ and $m = 2$ of Ramanujan's congruences were proved by himself:

$$p(5n+4) \equiv 0 \pmod{5}, \quad p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Modular forms and Ramanujan congruences

The proof of the Ramanujan congruences for general m requires us to go beyond mere power series manipulations and simple complex analysis. It requires modular forms, in particular, if $\ell = 11$.

Modular forms

The framework for modular forms

Modular forms and their variations are specific kinds of functions on the Poincaré upper half plane:

$$\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}.$$

This was the reason why we reformulated the generating series of $p(n)$ in terms of $e(n\tau) = \exp(2\pi i\, n\tau)$.

The variable τ allows us to express symmetries of functions of τ that are impossible to express for functions of $e(\tau)$.

The framework for modular forms

We have an action of $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathbb{H}$:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau = \frac{a\tau + b}{c\tau + d}.$$

One associates to this a very specific type of action on functions $\mathbb{H} \to \mathbb{C}$, the slash actions for weights $k \in \mathbb{Z}$:

$$\Big(f\big|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\Big)(\tau) := (c\tau + d)^{-k} f\Big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau\Big).$$

The framework for modular forms

For any group Γ with a map $\Gamma \to \mathrm{SL}_2(\mathbb{R})$, we obtain a corresponding action on $\mathbb{H}$. This is for example true for

$$\mathrm{Mp}_1(\mathbb{R}) := \big\{(\gamma,\omega) : \gamma \in \mathrm{SL}_2(\mathbb{R}), \\ \omega : \mathbb{H} \to \mathbb{C} \text{ holomorphic}, \omega(\tau)^2 = c\tau + d\big\}.$$

This allows for a slightly more general action on function $\mathbb{H} \to \mathbb{C}$, for weights $k \in \frac{1}{2}\mathbb{Z}$.

$$\big(f\big|_k (\gamma,\omega)\big)(\tau) := \omega(\tau)^{-2k} f(\gamma\tau).$$

The definition of modular forms

Consider a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ and $k \in \frac{1}{2}\mathbb{Z}$.
Let $\Gamma \subseteq \mathrm{Mp}_1(\mathbb{Z})$ be a subgroup of finite index, and $\chi : \Gamma \to \mathbb{C}^\times$ be a character. We say that f is a modular form of weight k for χ on Γ, if

$$\forall \gamma \in \Gamma : \; f\big|_k(\gamma,\omega) = \chi((\gamma,\omega))f$$

and

$$\exists a \in \mathbb{R}\, \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \; (f \circ \gamma)(\tau) = \mathcal{O}(\mathrm{Im}(\tau)^a) \text{ as } \mathrm{Im}(\tau) \to \infty.$$

Weakly holomorphic modular forms

When replacing the last condition by

$$\exists a \in \mathbb{R} \, \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \, (f \circ \gamma)(\tau) = \mathcal{O}(\exp(a \mathrm{Im}(\tau))) \text{ as } \mathrm{Im}(\tau) \to \infty,$$

we obtain the corresponding notion of weakly holomorphic modular forms.

Modular forms are covariants

Modular forms are nothing but covariants for specific actions of $\mathrm{Mp}_1(\mathbb{Z})$ or its subgroups on spaces of holomorphic functions $\mathbb{H} \to \mathbb{C}$ with a specific growth condition, say, the "moderate" one:

$$\mathrm{Hom}_\Gamma\big(\chi, \mathscr{H}(\mathbb{H})_k^{\mathrm{mod}}\big).$$

This point of view is unconventional, but will play a role later.

Existence of congruences

The partition generating function is modular

For our purpose, the prime example of weakly holomorphic modular forms are

$$\eta(\tau)^{-1} = e\big(\tfrac{-\tau}{24}\big) \sum_{n=0}^{\infty} p(n) e(n\tau) \quad \text{and}$$

$$e\big(\tfrac{-\tau}{24}\big) \sum_{\substack{n=0 \\ n \equiv b \pmod{a}}}^{\infty} p(n) e(n\tau), \quad a \in \mathbb{Z}_{\geq 1}, b \in \mathbb{Z}.$$

They both have weight $-\frac{1}{2}$.

The first one is on $\mathrm{Mp}_1(\mathbb{Z})$ for a generator of the character group $\mathrm{Mp}_1(\mathbb{Z})/[\mathrm{Mp}_1(\mathbb{Z}), \mathrm{Mp}_1(\mathbb{Z})]$, which has size 24.

Modular forms and Ramanujan congruences

Ramanujan's proof of the cases $m = 1$ and $m = 2$ was essentially based on power series manipulations; not using modular properties.

The cases $\ell \in \{5, 7\}$ and $m \geq 3$ where published by Waston in 1938, and he did use modular properties. Among Ramanujan's notes a discussion of these cases was later found.

Modular forms and Ramanujan congruences

Atkin proved the remaining cases $\ell = 11$, $m \geq 2$ of Ramanujan's conjecture in 1966, employing the modular forms

$$e\left(\frac{-\tau}{24}\right) \sum_{\substack{n=0 \\ 24n \equiv 1 \pmod{\ell^m}}}^{\infty} p(n) e(n\tau).$$

The key was a specific "Hecke" operator U_ℓ

$$\mathrm{U}_\ell \sum_{n \in \mathbb{Z}} c(n) e(n\tau) := \sum_{n \in \mathbb{Z}} c(\ell n) e(n\tau),$$

which has very good properties on modular forms with Fourier coefficients modulo a power of ℓ.

Ramanujan-type congruences

Atkin did more, and continued MacMahon's legacy. He explored the possibility of further divisibility patterns satisfied by $p(n)$ using state-of-the-art high performance computing.

What he found was a series of congruences of the type

$$p(\ell q^m n + b) \equiv 0 \pmod{\ell}, \quad \ell \neq q \text{ prime}, b \in \mathbb{Z}.$$

Multiplicative congruences

Atkin also observed that his new congruences come in families with $h^2(24b+1) \equiv (24b'+1) \pmod{\ell q^m}$, $h \in \mathbb{Z}$. The Ramanujan congruences correspond to $24b+1 \equiv 0 \pmod{\ell}$, so this could not be observed previously.

He proved some of them, for instance,

$$\begin{aligned} \forall n \in \mathbb{Z} &: p(11^3 13n + 237) \equiv 0 \pmod{13} \\ \forall n \in \mathbb{Z}, \left(\frac{n}{11}\right) = 1, \left(\frac{n}{13}\right) = -1 &: p(11^2 n) \equiv 0 \pmod{13}. \end{aligned}$$

This depends on further Hecke operators T_q, which always behave well, but he also needed a specific property that only seemed to hold for very few ℓ and q.

Infinitely many Ramanujan-type congruences

It took until 2000, when a paper by Ono appeared asserting:
Let $\ell \geq 5$ be a prime and m a positive integer. A positive proportion of the primes q have the property that

$$\forall n \in \mathbb{Z} \setminus q\mathbb{Z} : p\left(\frac{\ell^k q^3 n + 1}{24}\right) \equiv 0 \pmod{\ell}.$$

This yields infinitely many Ramanujan-type congruences on arithmetic progressions $\ell^k q^4 \mathbb{Z} + b$.

The phrase "positive proportion" has to be read as: The lower density of the set of such primes q is positive for each ℓ and m.

Infinitely many Ramanujan-type congruences

Behind this results were two new tools.

In the 70ies a theory of Galois representations associated with modular forms was developed by, among others, Eichler, Shimura, Katz, Deligne, and Serre. It gives reliable properties of the T_q for a "positive proportion" of q's. But it only applies to modular forms, not weakly holomorphic modular forms.

The weakly holomorphic modular form

$$e\Big(\frac{-\tau}{24}\Big) \sum_{\substack{n \in \mathbb{Z} \\ 24n \equiv 1 \pmod{\ell}}} p(n) e(n\tau)$$

has Fourier coefficients congruent to those of a cusp form; it is "congruent to a cusp form".

Obstruction of congruences

Obstructions to Ramanujan-type congruences

From a state where congruences were rare, the table had turned. The question was, when do these congruences not occur. Already, the case of Ramanujan congruences was elusive.

In 2003 Ahlgren and Boylan, based on work of Kiming and Olsson, showed that if for a prime ℓ and $b \in \mathbb{Z}$

$$\forall n \in \mathbb{Z} : p(\ell n + b) \equiv 0 \pmod{\ell},$$

then $\ell \in \{5, 7, 11\}$.

The structure that entered this time, was an interplay between the previous operators U_ℓ and a differential operator θ: Tate's θ-cycles from the 70ies.

Obstructions to Ramanujan-type congruences

Also the square classes of b in the arithmetic progression $\ell q^m \mathbb{Z} + b$ associated with a Ramanujan-type were the subject of investigation. Ahlgren and Oho had conjectured that

$$\left(\frac{1-24b}{\ell}\right) \neq 1.$$

This was proved by Radu in 2014. It required a resurgence of Atkin's multiplicative congruences.

Scarcity of Ramanujan-type congruences

By Ahlgren-Boyan, we know that there is only three primes ℓ for which

$$\exists b \in \mathbb{Z}\, \forall n \in \mathbb{Z} : p(\ell n + b) \equiv 0 \pmod{\ell}.$$

We know by Ono that there are infinitely many ℓ and q such that

$$\exists b \in \mathbb{Z}\, \forall n \in \mathbb{Z} : p(\ell q^4 n + b) \equiv 0 \pmod{\ell}.$$

If $m \in \{1, 2, 3\}$, how many primes ℓ and q are there such that

$$\exists b \in \mathbb{Z}\, \forall n \in \mathbb{Z} : p(\ell q^m n + b) \equiv 0 \pmod{\ell}?$$

Scarcity of Ramanujan-type congruences

In 2020, Ahlgren, Beckwith, and R. showed: Let $\ell \geq 5$ be a prime and $\delta \in \{0, -1\}$. Let Q be the set of primes q such that

$$\exists b \in \mathbb{Z}, \left(\frac{1-24b}{\ell}\right) = \delta \, \forall n \in \mathbb{Z} : p(\ell q n + b) \equiv 0 \pmod{\ell}.$$

Then Q has density zero or

$$\#\left\{n \leq X : \left(\frac{1-24n}{\ell}\right) = \delta, p(n) \not\equiv 0 \pmod{\ell}\right\} \ll \sqrt{X} \log X.$$

This required Montgomery's large sieve from the 60ies. It had recently shown up in work by Soundararajan on the question of how often $p(n) \not\equiv 0 \pmod{\ell}$.

Scarcity of Ramanujan-type congruences

A similar but more technical statement is available for

$$\exists b \in \mathbb{Z}\, \forall n \in \mathbb{Z} : p(\ell q^2 n + b) \equiv 0 \pmod{\ell}.$$

Computer calculations assert that there is no such congruence for

$$17 \leq \ell \leq 1000,\ 5 \leq q \leq 10.000,$$

so if there is any, they occur very late.

Atkin's case of third powers

There is one missing case, namely, it is not yet clear whether there are infinitely many ℓ and q such that

$$\exists b \in \mathbb{Z}\, \forall n \in \mathbb{Z} : p(\ell q^3 n + b) \equiv 0 \pmod{\ell}.$$

Aktin provided finitely many cases, but this hinged on a coincidental congruence satisfied by Hecke eigenvalues.

Work in progress by Ahlgren and Allen tries to elucidate this case.

Origins of Ramanujan-type congruences

All known Ramanujan-type congruences can be explained through Hecke operators U_ℓ or T_q. Is that a historical coincidence, or is there an actual connection?

For the partition function, we do not know yet, but we can treat the r-colored partitions for even r. The function $p_r(n)$ records the number of r-tuples of nonincreasing sequences of nonnegative integers that together sum up to n.

The generating series

$$e\Big(\frac{-r\tau}{24}\Big) \sum_{n=0}^{\infty} p_r(n) e(n\tau) = e\Big(\frac{-r\tau}{24}\Big) \prod_{n=1}^{\infty} \Big(\frac{1}{1 - e(n\tau)}\Big)^r$$

is a modular form of weight $-\frac{r}{2}$.

Origins of Ramanujan-type congruences

In 2020 R. proved: Let $\ell, q \geq 5$ be primes with $\ell \neq q$, m a positive integer, and $b \in \mathbb{Z}$. Suppose that

$$\left(\frac{r - 24b}{\ell}\right) \neq \left(\frac{r}{\ell}\right) \quad \text{and} \quad \forall n \in \mathbb{Z} : p_r(\ell q^m n + b) \equiv 0 \pmod{\ell}.$$

Then

$$\sum_{\substack{n=0 \\ n \equiv b \pmod{\ell}}}^{\infty} p_r(n) e\big((24n - r)\tau\big)$$

is a sum of generalized T_q-eigenforms modulo ℓ whose Fourier coefficients satisfy corresponding congruences.

This required the perspective on modular forms as covariants for $\mathrm{Mp}_1(\mathbb{Z})$, which allowed modular representation theory to enter.

Origins of Ramanujan-type congruences

In particular, with $q^{m'} := \gcd(q^m, 24b - r)$ we have the much stronger congruences

$$\forall n \in \mathbb{Z} \setminus q\mathbb{Z}, q^{m'} n \equiv 24b - r \pmod{\ell} :$$
$$p_r\left(\frac{\ell q^{m'} n + r}{24}\right) \equiv 0 \pmod{\ell} \equiv 0 \pmod{\ell}$$

extending the original one

$$\forall n \in \mathbb{Z} : p_r(q^m n + b) \equiv 0 \pmod{\ell} \equiv 0 \pmod{\ell}.$$

Origins of Ramanujan-type congruences

Another consequence concerns the density of q's. Given $\ell \geq 5$ and m, the set Q of primes with

$$\exists b \in \mathbb{Z}, \left(\frac{r-24b}{\ell}\right) \forall n \in \mathbb{Z} : p_r\left(\ell q^m n + b\right) \equiv 0 \pmod{\ell}$$

has a (positive) density.