Mathieu Moonshine for Siegel Modular Forms

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with Sutapa Samanta, arXiv:2011.07922 [hep-th]

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This talk is dedicated to the memory of the mathemagician

John McKay (18 November, 1939 – 19 April, 2022)

I have encountered his work during the course of my work on higher-dimensional partitions and Mathieu moonshine. Sadly, I have never met him but have corresponded with him by email.

Plan of Talk

Introduction

Mathieu Moonshine
- Multiplicative eta products
- Siegel Modular Forms
- Jacobi Forms

A product formula for $\Phi^{\rho}(\mathbf{Z})$

Concluding Remarks

Introduction

The Mathieu group M_{24}

- Discovered around 1861 by Mathieu. It has order 244823040.
- M_{24} can be realised as a permutation group acting on 24 points.
- It has 26 irreps with dimensions $23, 45, \overline{45}, 231, \overline{231}, \ldots$.
- Recall that conjugacy classes of permutation groups are represented by cycle shapes $\rho = 1^{a_1} 2^{a_2} 3^{a_3} \cdots N^{a_N}$ with $\sum_i i a_i = N$ for S_N.
- Conway and Norton observed that conjugacy classes of S_{24} that reduce to conj. classes of M_{24} have balanced cycle shapes.
- A cycle shape ρ with cycles $m_1 \leq m_2 \leq \cdots \leq m_t$, is said to be balanced if there exists a positive integer M (the balancing factor) such that it is invariant under the map $m_i \to M/m_i$.
- For example, $\rho = 1^6 2^9$ is unbalanced while $\rho = 1^8 2^8$ is balanced with $M = 2$ while $\rho = 2^{12}$ is balanced with $M = 4$.
- Not all balanced cycles shapes (of S_{24}) give rise to conjugacy classes of M_{24}. For example $\rho = 24^1$ does not occur in M_{24}.

Class Functions for Finite Groups

Definition

A function. $f : G \to \mathbb{C}$, is a class function if it is constant on a conjugacy class i.e.,

$$f(g) = f(h \cdot g \cdot h^{-1}) \ , \quad \forall \ g, h \in G \ .$$

- Let $\mathbb{V}$ be a G-module corresponding to an irreducible represention (irrep) r of G. Then, the character $\chi_r(g) := \mathrm{Tr}_V(g)$, is a class function.
- Let $r_1, r_2, \ldots, r_s$ denote all irreps of G. Then, for a class function f, one has

$$f^g = \sum_{i=1}^{s} a_i \, \chi_{r_i}(g) \ , \quad a_i \in \mathbb{C} \ .$$

- Let $\mathbb{V}$ be a reducible G-module. Then, $\mathrm{Tr}_{\mathbb{V}}(\,)$ is a class function with integral ceofficients $a_i \in \mathbb{Z}_{\geq 0}$.

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Class Functions for Finite Groups

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- Sometimes the G-module is graded by a set of integers. For instance, one may have $\mathbb{V} = \oplus_m V_m$ which is graded by a single integer. Then, one can define a class function with a variable (analog of fugacity in statistical physics)

$$f^g(\tau) = \sum_{i=1}^{s} \left(\sum_m a_{i,m} \ q^m \right) \chi_i(g) \ , \quad q = \exp(2\pi i \tau) \ .$$

What is moonshine?

> *Moonshine is not a well defined term, but everyone in the area recognizes it when they see it. Roughly speaking, it means weird connections between modular forms and sporadic simple groups.* – *R. E. Borcherds*

The first and most famous example of moonshine arose from the observation of John McKay (and extended by Norton) that the dimensions of the irreps of the Monster group appeared in the q-series modular j-function

$$j(\tau) - 744 = q^{-1} + [196883 + 1]\, q + [21296876 + 196883 + 1]\, q^2 + \cdots$$

A statement of Monstrous moonshine is that there exists a class function, $T^g(\tau)$, for $g \in M$, is a modular form of a suitable genus zero sub-group of $SL(2, \mathbb{R})$.

[Conway-Norton]

Mathieu moonshines

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Mathieu moonshines

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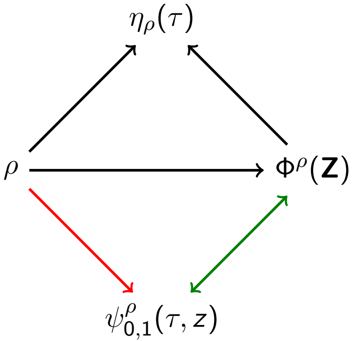
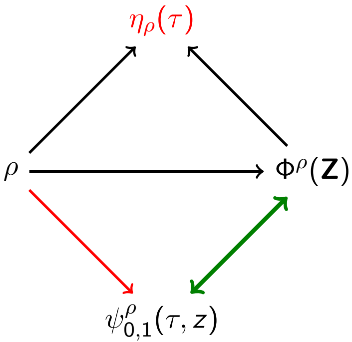
1. $\eta_\rho(\tau)$ is a multiplicative eta product,
2. $\psi^\rho_{0,1}(\tau, z)$ is a Jacobi form,
3. $\Phi^\rho(\mathbf{Z})$ is a genus-two Siegel modular form,

and their connections.



Multiplicative Eta Products



M_{24} and eta products

Theorem (Dummit-Kisilevsky-McKay and Mason, 1985)

To every conjugacy class, $\rho = 1^{a_1}2^{a_2}\cdots N^{a_N}$, of M_{24}, there exists a multiplicative modular form $\eta_\rho(\tau)$ given by the map

$$\rho = 1^{a_1}2^{a_2}\cdots N^{a_N} \longrightarrow \eta_\rho(\tau) := \prod_{j=1}^{N} \eta(j\tau)^{a_j} \ .$$

- A function $f(\tau) = \sum_{m=1}^{\infty} a_m q^m$ is multiplicative if $a_{mn} = a_m a_n$ for all (m, n) co-prime.
- One obtains $\eta_{1^{24}}(\tau) = \Delta(\tau)$, the discriminant function.
- Let p_ρ denote the smallest cycle in ρ and N the largest cycle. Then, $\eta_\rho(\tau)^{p_\rho}$ is a modular form of $\Gamma_0(N)$. [Cheng-Duncan, 2012]
- There exists a M_{24} class function which evaluates to $\eta_\rho(\tau)$ for all conjugacy classes ρ of M_{24}.

Is there a natural M_{24} module?

- It turns out that it is better to study $\eta_\rho(\tau)^{-1}$. For $\rho = 1^{24}$, consider

$$\frac{1}{\Delta(\tau)} = \sum_{m=0}^{\infty} p_{24}(m)\, q^{m-1} = q^{-1} + (1 + 23) + \cdots$$

  where $p_{24}(n)$ is the number of partitions of n with 24 colours.
- This appears as the Virasoro character of a trivial Verma module with central charge $c = 24$.
- It is also the partition function of non-zero (oscillator) modes of 24 chiral bosons. M_{24} is the sub-group of the S_{24} that naturally permutes these 24 bosons.
- However, there is natural reduction of S_{24} to M_{24}. It remains an open problem to construct a module on which M_{24} acts naturally. (No analog of what Frenkel-Lepowsky-Meurman did for Monstrous moonshine.)

These eta products count half-BPS states!

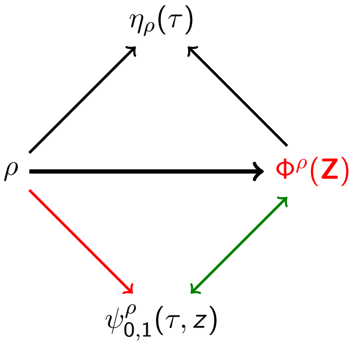
- Consider 4d string theory with $\mathcal{N}=4$ supersymmetry obtained from compactifying type II string theory on $K3\times T^2$.
- In the type IIA frame, there are a class of electrically charged $\frac{1}{2}$-BPS states given by excited states of the solitonic heterotic string with the supersymmetric right movers in the ground state and the left movers at level $(n+1)$.
- These have electric charge $\frac{1}{2}\mathbf{q}_e^2 = n$. Let $d(n)$ denote their microscopic degeneracy. Then, one has (with $q=\exp(2\pi i\tau)$)

$$\frac{16}{\eta(\tau)^{24}} = \sum_{n=-1}^{\infty} d(n)\ q^n \ ,$$

- Let g be a finite symplectic automorphism of $K3$. Then, g is an element of a subgroup of $M_{23}\subset M_{24}$ with order ≤ 8. [Mukai]
- Then, the multiplicative eta products generate $\frac{1}{2}$-BPS states twined by g. Let $\rho=[g]$. [GopalaKrishna-SG, 2009]

$$\frac{16}{\eta_\rho(\tau)} = \sum_{n=-1}^{\infty} d^g(n)\ q^n \ ,$$

Siegel Modular Forms



Counting quarter BPS states in string theory

- Similarly, let $D(n,\ell,m)$ denote the microscopic degeneracy of dyonic $\frac{1}{4}$-BPS states with charges $(\frac{1}{2}\mathbf{q}_e^2, \mathbf{q}_e\cdot\mathbf{q}_m, \frac{1}{2}\mathbf{q}_m^2) = (n,\ell,m)$.
- In 1996, Dijkgraaf-Verlinde-Verlinde conjectured that generating function was given by weight ten Igusa cusp form of $Sp(2,\mathbb{Z})$.
$$\frac{64}{\Phi_{10}(\mathbf{Z})} = \sum_{(n,\ell,m)} D(n,\ell,m)\ q^n r^\ell s^m\ ,$$
where $\mathbf{Z} = \left(\begin{smallmatrix}\tau & z\\ z & \tau'\end{smallmatrix}\right)$, $r = \exp(2\pi i z)$ and $s = \exp(2\pi i \tau')$.
- This was the first example where, in the limit of large charges, the degeneracy matched the one given by the entropy of four-dimensional black holes. The famous matching of Strominger-Vafa was for a five-dimensional black hole.
- Let g be a symplectic automorphism of K3. Then, let
$$\frac{64}{\Phi^\rho(\mathbf{Z})} = \sum_{(n,\ell,m)} D^g(n,\ell,m)\ q^n r^\ell s^m\ ,$$

Mathieu moonshine

Ex-Conjecture (SG-Gopala Krishna, 2009)

To every M_{24}-conjugacy class, there exists a Siegel Modular form, $\Phi^\rho(\mathbf{Z})$, of a level N sub-group of $Sp(2,\mathbb{Z})$ of suitable weight. There exists a class function that generates all these modular forms.

- The conjecture was not stated explicitly as mentioned above but stated in the form a map:

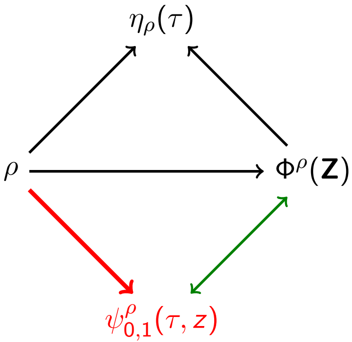
$$\rho \longrightarrow \Phi^\rho(\mathbf{Z}) \ ,$$

  along with a small number of examples arising from symplectic automorphisms.
- The Siegel Modular forms were constructed via an additive lift.

$$\rho \to \eta_\rho(\tau) \xrightarrow{\textit{additive lift}} \Phi^\rho(\mathbf{Z}) \ ,$$

- However, the additive lift works in a limited number of examples.

Jacobi Forms



The elliptic genus of hyper-Kähler manifolds

- The elliptic genus of compact hyper-Kähler manifolds (of real dim $4k$) can be expanded in terms of characters of the $\mathcal{N} = 4$ SCA at level k. They are Jacobi forms of weight zero and index k, [Eguchi-Hikami]
- In particular, one has

$$\chi(M;\tau,z) = \chi(M)\ \mathcal{C}_k(\tau,z) + \sum_{a=1}^{k} \Sigma^{(a)}(\tau)\ \mathcal{B}_k^{(a)}(\tau,z)\ ,$$

  where $\mathcal{C}_k(\tau,z)$ is a massless Ramond character and $\mathcal{B}_k^{(a)}(\tau,z)$ are massive Ramond characters.
- One has $\mathcal{C}_k(\tau,0) = 1$ and $\mathcal{B}_k^{(a)}(\tau,0) = 0$. Thus $\chi(M;\tau,0) = \chi(M)$. The elliptic genus is thus a two-variable extension of the Euler characteristic of M.
- For $K3$, which has complex dimension 2, one obtains a index one Jacobi form.

Mathieu moonshine

- When $M = K3$, the elliptic genus of $K3$ can be written as

$$\psi_{0,1}(\tau,z) = \mathcal{E}(K3;\tau,z) = 24\ \mathcal{C}_1(\tau,z) + \overbrace{q^{-1/8}\ \Sigma(\tau)}^{\text{mock modular form}}\ \mathcal{B}_1(\tau,z)$$

- The function $\Sigma(\tau)$ has the following Fourier expansion

$$\Sigma(\tau) = \left(\beta + \sum_{n=1}^{\infty} A(n)\ q^n\right), \\ = 2(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \cdots)$$

- The numbers in red are the dimensions of the irreps of M_{24}.

[Eguchi-Ooguri-Tachikawa].

- Is there a class function with the following q-expansion?

$$\Sigma^\rho(\tau) = -2 + (\chi_{45} + \chi_{\overline{45}})\ q + (\chi_{231} + \chi_{\overline{231}})\ q^2 + \cdots$$

Ex-Conjecture (Eguchi-Ooguri-Tachikawa,2010)

There exists a class function that associates a weight zero, index 1, Jacobi form to each conjugacy class of M_{24}.

- The twining elliptic genera are Jacobi forms at level M. Recall that M is the balancing factor of the cycle shape.
- At each level, the space of Jacobi forms is of some finite dimension. For prime levels, it is two-dimensional.
- In other words, the first few terms in the character expansion uniquely determines the Jacobi form! The rest of the terms provide validation.
- Using this idea, Jacobi forms have been determined for all 26 conjugacy classes of M_{24}. [Cheng,Gaberdiel-Hohenegger-Volpato,Eguchi-Hikami]
- The rewriting of the elliptic genus of K3 as a class function has been carried out to order q^{1000}. [Eguchi-Hikami]
- In 2012, Terry Gannon proved that such a class function exists to all orders if it verified to order q^{400}.

Is there a M_{24} module $\mathcal{K}$ such that $\psi_{0,1}(\tau, z)$ arises as a trace function?

- There doesn't appear to be one. Gannon's proof is one of existence rather than a constructive one.
- There is no point in the moduli space of the CFT for the non-linear sigma model with target space $K3$ where the full group M_{24} appears as a symmetry.
  [Gaberdiel-Hohenegger-Volpato]
- There is some interesting work by Taormina and Wendland which attempts to obtain M_{24} by a process that they call "symmetry surfing" by combining symmetries at different points in the moduli space of the CFT.
- Gaberdiel-Taormina-Volpato-Wendland have constructed a CFT which exhibits M_{20} as a symmetry.
- Nevertheless, assuming that such a module $\mathcal{K}$ exists is useful in making progress as we will see.

A product formula for $\Phi^{\rho}(\mathbf{Z})$

Conjecture (SG, 2011)

There is a graded M_{24}-module $V^\natural = \oplus_{(n,\ell,m)} V_{(n,\ell,m)}$ such that the Igusa cusp form

$$\frac{1}{\Phi_{10}(\mathbf{Z})} = \sum_{(n,\ell,m)} dim\left[V_{(n,\ell,m)}\right] q^n r^\ell s^m \ ,$$

and one twining by $g \in M_{24}$ leads to Siegel modular forms

$$\frac{1}{\Phi^\rho(\mathbf{Z})} = \sum_{(n,\ell,m)} Tr\left[g\big|_{V_{(n,\ell,m)}}\right] q^n r^\ell s^m \ .$$

(ρ is the conjuacy class of $g \in M_{24}$)

In the rest of the talk, we will prove the existence of a class function, $\Phi^\rho(\mathbf{Z})$, that becomes a Siegel modular form at level N. However, we do not have a direct construction of the module.

Breaking up $V^\natural$

- The physical setting provides a break up of $V^\natural$. The Igusa cusp form naturally splits into three parts in string theory:

$$\frac{64}{\Phi_{10}(\mathbf{Z})} = \underbrace{\left[\frac{4\ \eta(\tau)^6}{\theta_1(\tau,z)^2}\right]}_{\text{(i)}} \times \underbrace{\left[\frac{16}{\eta(\tau)^{24}}\right]}_{\text{(ii)}} \times \underbrace{\left[\frac{1}{\mathcal{E}(K3;\mathbf{Z})}\right]}_{\text{(iii)}} ,$$

- $V^\natural = W \otimes \mathcal{B} \otimes \mathcal{S}$ with

$$\sum_{(n,\ell)} \mathrm{sdim}\left[W_{(n,\ell,0)}\right] q^n r^\ell s^0 = \frac{\eta(\tau)^6}{\theta_1(\tau,z)^2} ,$$

$$\sum_{n} \mathrm{sdim}\left[\mathcal{B}_{(n,0,0)}\right] q^n r^0 s^0 = \frac{1}{\eta(\tau)^{24}} ,$$

$$\sum_{(n,\ell,m)} \mathrm{sdim}\left[\mathcal{S}_{(n,\ell,m)}\right] q^n r^\ell s^m = \frac{1}{\mathcal{E}(K3,\mathbf{Z})} ,$$

where $\mathcal{E}(K3;\mathbf{Z})$ is the second-quantized elliptic genus of $K3$

The modules W and $\mathcal{B}$

- W is associated with the motion of $D1 - D5$ branes in TN space. It is defined to be the BPS subspace of the Hilbert space of the associated SCFT.
- This SCFT does not see dynamics associated with $K3$ and hence will not be acted on by any K3-related symmetries. It is thus a trivial M_{24}-module.
- $\mathcal{B}$ is nothing but the Fock space of 24 chiral bosons that appears in half-BPS counting. As discussed there, it is a S_{24}-module and hence a M_{24} module.
- In particular, we already know that

$$\sum_n \mathrm{sTr}\left[g\big|_{\mathcal{B}_{(n,0,0)}} \right] q^n r^0 s^0 = \frac{1}{\eta_\rho(\tau)}$$

  This connects up with the moonshine associated with multiplicative eta projects.
- We will next see how the elliptic genus of K3 naturally determines the second quantized elliptic genus of K3.

The second-quantized elliptic genus of K3

- Define the second-quantized elliptic genus of $K3$ to be

[Dijkgraaf-Moore-Verlinde-Verlinde]

$$\frac{1}{\mathcal{E}(K3;\mathbf{Z})} := s^{-1} \times \left(1 + \sum_{m=1}^{\infty} s^m \, \chi(S^m(K3);\tau,z)\right) ,$$

where $S^m(K3) = K3^{\times m}/S_m$ refers to the m-th symmetric product of $K3$.

- The elliptic genus of symmetric powers of K3 can be obtained by computing the elliptic genus in the permutation orbifold CFT. For instance, one has

$$\chi(S^2(K3);\tau,z) = \underbrace{\tfrac{1}{2}\left[\chi(K3,\tau,z)^2 + \chi(K3,2\tau,2z)\right]}_{\text{untwisted sector}} + \underbrace{\tfrac{1}{2}\left[\chi(K3,\tfrac{\tau}{2},z) + \chi(K3,\tfrac{\tau+1}{2},z)\right]}_{\text{twisted sector}}$$

- DMVV rewrite the above 'replication formula' as follows:

$$\chi(S^2(K3);\tau,z) = \tfrac{1}{2}(\chi(K3,\tau,z))^2 + \tfrac{1}{2}\sum_{ad=2}\sum_{b=0}^{d-1}\chi(K3,\tfrac{a\tau+b}{d},az)$$

A product formula for the second-quantized elliptic genus

- Replication formulae occur for all cases leading to the following formula for the second-quantized elliptic genus.

$$\frac{1}{\mathcal{E}(K3;\mathbf{Z})} = \exp\left(- \sum_{m=1}^{\infty} s^m \ T(m) \cdot \psi_{0,1}(\tau, z) \right) ,$$

where $T(m)$ is the Hecke-like operator

$$T(m) \cdot \psi_{0,1}(\tau, z) \equiv \frac{1}{m} \sum_{ad=m} \sum_{b=0}^{d-1} \psi_{0,1}\left(\tfrac{a\tau+b}{d}, az\right) .$$

- This leads to a Borcherds type product formula

$$\frac{1}{\mathcal{E}(K3;\mathbf{Z})} = s^{-1} \times \prod_{\substack{n\geq 0 \\ m>0,\ell}} \frac{1}{(1-q^n r^\ell s^m)^{c(nm,\ell)}} ,$$

where $c(nm,\ell)$ are the Fourier-Jacobi coefficients of the elliptic genus of $K3$:

$$\psi_{0,1}(\tau, z) := \chi(K3;\tau, z) = \sum_{n\geq 0,\ell} c(n,\ell)\ q^n r^\ell .$$

A product formula for twined second-quantized elliptic genus

- Let $g \in M_{24}$ and ρ its conjugacy class.
- We obtain a similar formula for the twined version [SG, 2011]

$$\frac{1}{\mathcal{E}^{\rho}(K3;\mathbf{Z})} = \exp\Big(- \sum_{m=1}^{\infty} s^m \ \hat{T}(m) \cdot \psi_{0,1}(\tau, z)\Big) \ ,$$

where $\hat{T}(m)$ is a twisted Hecke-like operator

$$\hat{T}(m) \cdot \psi_{0,1}^{[g]}(\tau, z) \equiv \frac{1}{m} \sum_{ad=m} \sum_{b=0}^{d-1} \ \psi_{0,1}^{[g^a]} \left(\tfrac{a\tau+b}{d}, az\right) \ .$$

- This again leads to a Borcherds type product formula given in terms of the Fourier-Jacobi coefficients of multiple twined elliptic genera, $\psi_{0,1}^{[g^a]}(\tau, z)$ for $a = 0, 1, \ldots, \mathrm{order}(g) - 1$.
- Similar results were obtained independently by Cheng and Duncan.

Modularity of the product formula

- The product formulae for the twining genera is very nice and immediately leads to product formulae for $\Phi^{\rho}(\mathbf{Z})$.
- Note that the product formula works for all 26 conjugacy classes of M_{24} modulo the assumption that $\mathcal{K}$ is a M_{24}-module – this was proved by Gannon a year later.
- However, nothing in the construction guarantees the modularity of the object.
- In 2011, I was able to match some but not all cases to standard constructions of Siegel modular forms in number theory.
- The additive lift does not work for weights ≤ 0. This is potentially the case for some of the conjugacy classes.
- Martin Raum argued that these product formulae could be understood as products of standard Siegel modular forms. However, he could show that it was true in some cases leaving the problem of proving modularity an open.
- This was solved in 2019 by my student Sutapa Samanta in her doctoral thesis.

Theorem (Cléry-Gritsenko, 2008)

Let ψ be a nearly holomorphic Jacobi form of weight 0 and index 1 of $\Gamma_0(N)$. Assume that for all cusps f/e of $\Gamma_0(N)$ one has $\frac{h_e}{N_e} c_{f/e}(n,\ell) \in \mathbb{Z}$ if $4n - \ell^2 \leq 0$. Then the product

$$B_\psi(\mathbf{Z}) = q^A r^B s^C \prod_{f/e \in \mathcal{P}} \prod_{\substack{n,\ell,m \in \mathbb{Z} \\ (n,\ell,m)>0}} \left(1 - (q^n r^\ell s^m)^{N_e}\right)^{\frac{h_e}{N_e} c_{f/e}(nm,\ell)},$$

where $A = \frac{1}{24} \sum_{\substack{f/e \in \mathcal{P} \\ \ell \in \mathbb{Z}}} h_e \, c_{f/e}(0,\ell)$, $B = \frac{1}{2} \sum_{\substack{f/e \in \mathcal{P} \\ \ell \in \mathbb{Z}_{>0}}} \ell h_e \, c_{f/e}(0,\ell)$, $C = \frac{1}{4} \sum_{\substack{f/e \in \mathcal{P} \\ \ell \in \mathbb{Z}}} \ell^2 h_e \, c_{f/e}(0,\ell)$,

defines a meromorphic Siegel modular form of weight

$$k = \frac{1}{2} \sum_{\substack{f/e \in \mathcal{P} \\ \ell \in \mathbb{Z}}} \frac{h_e}{N_e} c_{f/e}(0,0)$$

with respect to $\Gamma_1(N)^+$ possibly with character. The character is determined by the zeroth Fourier-Jacobi coefficient of $B_\psi(\mathbf{Z})$ which is a Jacobi form of weight k and index C of the Jacobi subgroup of $\Gamma_1(N)^+$.

This construction thus leads to a Siegel Modular Form: $B_\psi(\mathbf{Z})$

Modularity of $\Phi^\rho(\mathbf{Z})$

Proposition (Govindarajan-Samanta, 2019)

For $g \in M_{24}$, let $\rho_m = [g^m]$, $\rho = \rho_1$, $\psi^{\rho_m} = \psi^{\rho_m}_{0,1}(\tau, z)$ and $\Phi^\rho_k(\mathbf{Z})$ be defined by product formula implied by Mathieu moonshine and let $B_\psi(\mathbf{Z})$ be the Siegel modular form defined by the multiplicative lift given by Cléry-Gritsenko. Then

$$\left(\Phi^\rho(\mathbf{Z})\right)^{p_\rho} = \prod_{m|p_\rho} \left(B_{\frac{p_\rho}{m}\psi^{\rho_m}}(m\,\mathbf{Z})\right) \ ,$$

where p_ρ is the length of the shortest cycle in ρ.

- The proof is done one a case by case basis and appear in Sutapa's thesis.
- This shows that $\Phi^\rho(\mathbf{Z})$ are indeed Siegel modular forms of subgroups at level order(g) of the genus two symplectic group. They transform with phases that are p_ρ-th roots of unity.
- A different proof of modularity appears in the work of Persson-Volpato who show that these are Siegel modular forms at higher level M, where M is the balancing factor of the conjugacy class.

Concluding Remarks

- There is an analogous story for generalized Mathieu moonshine which involves pairs of commuting elements of M_{24}. All such pairs were classified long ago by Mason.
- I studied a few of them as a part of an attempt to connect to BKM Lie algebras.
- Persson and Volpato have obtained and proved modularity in all such cases.
- Cheng, Duncan and and Harvey discovered a generalization of Mathieu moonshine called umbral moonshine. There is one example associated with each Niemeier lattice in 24 dimensions. The structure is very rich and I do not understand all aspects in that story.
- For a subset of cases associated to A-type Niemeier lattices, Sutapa and I have constructed Siegel modular forms that are analogs of $\Phi_{10}(\mathbf{Z})$. The group generalize $Sp(2,\mathbb{Z})$ – these are called paramodular groups. The cases of twining remain open.
- Is there a similar situation for other cases of umbral moonshine?
- The big open question is about providing natural constructions for the modules $\mathcal{K}$ as well as $V^\natural$. (talk by Beneish on this channel!)

THANK YOU