Asymptotic equidistribution for partition statistics and topological invariants

joint work with William Craig and Joshua Males

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Motivation

A *partition* λ of a positive integer n is a list of non-increasing positive integers, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, that satisfies $|\lambda| := \lambda_1 + \cdots + \lambda_m = n$.

$$p(n) := \# \text{ of partitions of } n$$

Example

For $n = 4$ the possible partitions are given by

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

Thus we have $p(4) = 5$.

Motivation

Equidistribution properties of certain objects are a central theme studied by many authors in many mathemathical fields.

What do we mean when we say asymptotic equidistribution?

Suppose that $c(n)$ is an arithmetic counting function e.g. $c(n) = p(n)$. Suppose $s(\lambda)$ is an integer valued partition invariant and let

$$c(a, b; n) := \#\{\text{partitions of } n \ : \ s(\lambda) \equiv a \pmod{b}\}.$$

To say that equidistribution holds is to say that

$$c(a, b; n) \sim \frac{1}{b} c(n)$$

as $n \to \infty$.

Motivation

Examples for recently studied modular typed objects:

1. Asymptotic equidistribution of partition ranks (Males).
2. Asymptotic equidistribution results for partitions into k-th powers (Ciolan).
3. Asymptotic equidistribution for Hodge numbers and Betti numbers of certain Hilbert schemes of surfaces (Gillman–Gonzalez–Ono–Rolen–Schoenbauer).
4. Asymptotic equidistribution of partitions whose parts are values of a given polynomial (Zhou).

Motivation

Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ has a *Ferrers–Young diagram*:

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \ldots & \bullet & \leftarrow & \lambda_1 \text{ many nodes} \\ \bullet & \bullet & \ldots & \bullet & & \leftarrow & \lambda_2 \text{ many nodes} \\ \vdots & \vdots & \vdots & & & & \\ \bullet & \ldots & \bullet & & & \leftarrow & \lambda_m \text{ many nodes} \end{array}$$

The node in row k and column j has *hook length*

$$h(k,j) := (\lambda_k - k) + (\lambda'_j - j) + 1,$$

$$\lambda'_j := \# \text{ nodes in column } j.$$

Motivation

Let $\mathcal{H}_t(\lambda)$ denote the multiset of *t-hooks*, those hook lengths which are multiples of a fixed positive integer t, of a partition λ. Let

$$p_t^e(n) := \#\{\lambda \text{ a partition of } n : \#\mathcal{H}_t(\lambda) \text{ is even}\},$$
$$p_t^o(n) := \#\{\lambda \text{ a partition of } n : \#\mathcal{H}_t(\lambda) \text{ is odd}\}.$$

Craig–Pun:
For even t the partitions of n are asymptotically equidistributed between these two subsets, for odd t they are not.

Bringmann–Craig–Males–Ono:
On arithmetic progressions modulo odd primes t-hooks are not asymptotically equdistributed. The Betti numbers of two specific Hilbert schemes are asymptotically equdistributed.

Wright's Circle Method

Hardy–Ramanujan, 1918

$$p(n) \sim \frac{1}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}}, \qquad \text{as } n \to \infty.$$

The essence of Wright's method is to use Cauchy's theorem. We have

$$\mathcal{A}(\tau) := \sum_{n\geq 0} a(n)q^n \quad \longrightarrow \quad a(n) = \frac{1}{2\pi i}\int_C \frac{\mathcal{A}(q)}{q^{n+1}}dq,$$

where $q = e^{2\pi i\tau}$.

One then splits the integral into two arcs, the *major arc* and *minor arc*.

Following Wright and the work of Ngo–Rhoades, Bringmann–Craig–Males–Ono proved the following variant of Wright's Circle Method.

Variant of Wright's Circle Method

Let $M > 0$ be a fixed constant and $z = x + iy \in \mathbb{C}$, with $x > 0$ and $|y| < \pi$.

Consider the following hypotheses:

(i) As $z \to 0$ in the bounded cone $|y| \leq Mx$ (major arc), we have

$$F(e^{-z}) = z^B e^{\frac{A}{z}} \left(\alpha_0 + O_M(|z|)\right),$$

where $\alpha_0 \in \mathbb{C}$, $A \in \mathbb{R}^+$, and $B \in \mathbb{R}$.

(ii) As $z \to 0$ in the bounded cone $Mx \leq |y| < \pi$ (minor arc), we have

$$|F(e^{-z})| \ll_M e^{\frac{1}{\mathrm{Re}(z)}(A-\kappa)},$$

for some $\kappa \in \mathbb{R}^+$.

Variant of Wright's Circle Method

Bringmann–Craig–Males–Ono, 2021

Suppose that $F(q)$ is analytic for $q = e^{-z}$ where $z = x + iy \in \mathbb{C}$ satisfies $x > 0$ and $|y| < \pi$, and suppose that $F(q)$ has an expansion $F(q) = \sum_{n=0}^{\infty} c(n)q^n$ near 1.
If (i) and (ii) hold, then as $n \to \infty$ we have

$$c(n) = n^{\frac{1}{4}(-2B-3)} e^{2\sqrt{An}} \left(p_0 + O\left(n^{-\frac{1}{2}}\right)\right),$$

where $p_0 = \alpha_0 \dfrac{\sqrt{A}^{B+\frac{1}{2}}}{2\sqrt{\pi}}$.

Setting of Central Theorem

Let $q = e^{-z}$, where $z = x + iy \in \mathbb{C}$ with $x > 0$ and $|y| < \pi$.
Furthermore let $\zeta = \zeta_b^a := e^{\frac{2\pi i a}{b}}$ ($b \geq 2$ and $0 \leq a < b$).
Assume that we have a generating function on arithmetic progressions $a \pmod{b}$ given by

$$H(a, b; q) := \sum_{n \geq 0} c(a, b; n) q^n,$$

for some coefficients $c(a, b; n)$ such that

$$H(a, b; q) = \frac{1}{b} \sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j; q)$$

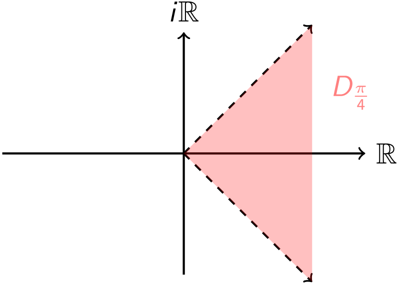
for some generating functions $H(\zeta; q)$, with

$$H(q) := H(1; q) = \sum_{n \geq 0} c(n) q^n.$$

Setting of Central Theorem

Let $H(a, b; q)$ and $H(\zeta; q)$ be analytic on $|q| < 1$ such that the above holds.
Let $C = C_n$ be a sequence of circles centered at the origin inside the unit disk with radii $r_n \to 1$ as $n \to \infty$ that loops around zero exactly once.
For $0 \leq \theta < \frac{\pi}{2}$ let

$$D_\theta := \left\{ z = re^{i\alpha} \colon r \geq 0 \text{ and } |\alpha| \leq \theta \right\}.$$



Setting of Central Theorem

For $\theta > 0$, let $\widetilde{C} := C \cap D_\theta$ and $C \setminus \widetilde{C}$ be arcs such that the following hypotheses hold.

(1) As $z \to 0$ outside of D_θ, we have

$$\sum_{j=1}^{b-1} \zeta_b^{-aj} H(\zeta_b^j; e^{-z}) = O\left(H(1; e^{-z})\right).$$

(2) As $z \to 0$ in D_θ, we have for each $1 \leq j \leq b-1$ that

$$H(\zeta_b^j; e^{-z}) = o\left(H(1; e^{-z})\right).$$

(3) As $n \to \infty$, we have

$$c(n) \sim \frac{1}{2\pi i} \int_{\widetilde{C}} \frac{H(1; q)}{q^{n+1}} dq.$$

Central Theorem

C.–Craig–Males, 2021

As $n \to \infty$, we have

$$c(a, b; n) \sim \frac{1}{b} c(n).$$

In particular, if $H(1; q)$ and $H(\zeta; q)$ satisfy the conditions of BCMO we have that

$$c(a, b; n) \sim \frac{1}{b} c(n) \sim \frac{1}{b} n^{\frac{1}{4}(-2B-3)} e^{2\sqrt{An}} \left(p_0 + O\left(n^{-\frac{1}{2}}\right)\right)$$

as $n \to \infty$.

Idea of the proof

1. Use Cauchy's theorem and the decomposition of $H(a, b; q)$ to obtain

$$c(a, b; n) = \frac{1}{b}\left[\frac{1}{2\pi i}\int_C \frac{\sum_{j=0}^{b-1}\zeta_b^{-aj} H(\zeta_b^j; q)}{q^{n+1}} dq\right].$$

2. Break down the integral over C into the components $\widetilde{C}$ and $C\backslash\widetilde{C}$ and look at each of them seperately.
3. Along $C\backslash\widetilde{C}$ we have by conditions (1) and (3) that as $n \to \infty$

$$\frac{1}{2\pi i}\int_{C\backslash\widetilde{C}} \frac{\sum_{j=0}^{b-1}\zeta_b^{-aj} H(\zeta_b^j; q)}{q^{n+1}} dq = o\left(\frac{1}{2\pi i}\int_{\widetilde{C}} \frac{H(1; q)}{q^{n+1}} dq\right).$$

Idea of the proof

4. On $\widetilde{C}$ we obtain with (2) and as $n \to \infty$ that

$$\frac{1}{2\pi i} \int_{\widetilde{C}} \frac{\sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j; q)}{q^{n+1}} dq \sim \frac{1}{2\pi i} \int_{\widetilde{C}} \frac{H(1; q)}{q^{n+1}} dq.$$

5. The first claim follows by combining the estimates along $\widetilde{C}$ and $C \setminus \widetilde{C}$.
6. If we assume $H(1; q)$ and $H(\zeta_b^j; q)$ satisfy the hypotheses of BCMO, then (1) – (3) are satisfied and the result follows by the asymptotic for $c(n)$ in BCMO. □

Asymptotic convexity

C.–Craig–Males, 2021

Let $0 \leq a < b$ and $b \geq 2$. Assume that $H(1;q)$ and $H(\zeta;q)$ satisfy the conditions of BCMO. Then for sufficiently large n_1, n_2 we have

$$c(a,b;n_1)c(a,b;n_2) > c(a,b;n_1+n_2).$$

Known examples:

1. partition function (Bessenrodt–Ono)
2. partition ranks congruent to $a \pmod{b}$ (Hou–Jagadeesan, Males)

Log-concavity

C.—Craig—Males, 2021

Let $0 \leq a < b$ and $b \geq 2$. Assume that $H(1;q)$ and $H(\zeta;q)$ satisfy the conditions of BCMO. For large enough n, we have

$$c(a,b;n)^2 \geq c(a,b;n-1)c(a,b;n+1).$$

Known examples:

1. partition function (DeSalvo—Pak)
2. unimodal sequences of size n and rank m (Bringmann—Jennings-Shaffer—Mahlburg—Rhoades)
3. spt-function (Dawsey—Masri)

Ramanujan congruences, 1921

For $n \geq 0$ we have

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Example

The *rank* of a partition λ is the largest part minus the number of parts.

The ranks of the partitions of 4:

partition	rank
(4)	$3 \equiv 3 \pmod{5}$
$(3,1)$	$1 \equiv 1 \pmod{5}$
$(2,2)$	$0 \equiv 0 \pmod{5}$
$(2,1,1)$	$-1 \equiv 4 \pmod{5}$
$(1,1,1,1)$	$-3 \equiv 2 \pmod{5}$

The rank

$N(a, b; n) :=$ # of partitions of n with rank congruent to $a \pmod b$

C.−Craig−Males, 2021

Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that

$$N(a, b; n) = \frac{1}{b} p(n) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

The equidistribution of $N(a, b; n)$ was already proven by Males in 2021 using Ingham's Tauberian theorem.

The crank

$$\text{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0 \end{cases}$$

$$\omega(\lambda) := \#\text{ of ones in } \lambda, \quad \mu(\lambda) := \#\text{ of parts greater than } \omega(\lambda)$$

$$M(a, b; n) := \#\text{ of partitions of } n \text{ with crank congruent to } a \pmod{b}$$

C.–Craig–Males, 2021

Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that

$$M(a, b; n) = \frac{1}{b} p(n) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

The first residual crank

An *overpartition* is a partition where the first occurrence of each distinct number may be overlined.

Example

The overpartitions of 4 are given by

$$(4),\ (\overline{4}),\ (3,1),\ (\overline{3},1),\ (3,\overline{1}),\ (\overline{3},\overline{1}),\ (2,2),\ (\overline{2},2),$$
$$(2,1,1),\ (\overline{2},1,1),\ (2,\overline{1},1),\ (\overline{2},\overline{1},1),\ (1,1,1,1),\ (\overline{1},1,1,1).$$

The *first residual crank* of an overpartition is given by the crank of the subpartition consisting of the non-overlined parts.

The first residual crank

Example

So the first residual crank of $(2, \overline{1}, 1)$ is given by the crank of $(2, 1)$ which equals 0.

$$\overline{M}(a, b; n) := \# \text{ of overpartitions of } n \\ \text{with first residual crank congruent to } a \pmod{b}$$

C.–Craig–Males, 2021

Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that

$$\overline{M}(a, b; n) = \frac{1}{8bn} e^{\pi\sqrt{n}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

Plane partitions

A *plane partition* of n is a two-dimensional array $\pi_{j,k}$ of non-negative integers $j, k \geq 1$, that is non-increasing in both variables, i.e., $\pi_{j,k} \geq \pi_{j+1,k}$, $\pi_{j,k} \geq \pi_{j,k+1}$ for all j and k, and fulfils $|\Lambda| := \sum_{j,k} \pi_{j,k} = n$.

$$\mathrm{pp}(n) := \#\text{ plane partitions of } n$$

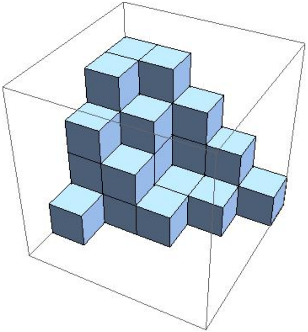
Example

For $n = 3$ we have the plane partitions:

$$\begin{matrix} 1 & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & \end{matrix} \qquad \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \qquad \begin{matrix} 2 & 1 \end{matrix} \qquad \begin{matrix} 2 \\ 1 \end{matrix} \qquad \begin{matrix} 3 \end{matrix}$$

Thus we have $\mathrm{pp}(3) = 6$.

Plane partitions



A plane partition may be represented visually by the placement of a stack of $\pi_{j,k}$ unit cubes above the point (j, k) in the plane, giving a three-dimensional solid.
The sum $|\Lambda|$ then describes the number of cubes of which the plane partition consists.

Plane partitions

Let $\Lambda = \{\pi_{j,k}\}_{j,k\geq 1}$ and define its *trace* by $t(\Lambda) = \sum_{j=1}^{\infty} \pi_{j,j}$.

$$\mathrm{pp}(a, b; n) := \#\{\Lambda : |\Lambda| = n, t(\Lambda) \equiv a \pmod{b}\}$$

Example

We have that $\mathrm{pp}(0, 2; 3) = 2$ and $\mathrm{pp}(1, 2; 3) = 4$.

C.–Craig–Males, 2021

Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that

$$\mathrm{pp}(a, b; n) \sim \frac{1}{b}\,\mathrm{pp}(n) \sim \frac{1}{b}\frac{\zeta(3)^{\frac{7}{36}}}{\sqrt{12\pi}}\left(\frac{n}{2}\right)^{-\frac{25}{36}} \exp\left(3\zeta(3)^{\frac{1}{3}}\left(\frac{n}{2}\right)^{\frac{2}{3}} + \zeta'(-1)\right).$$

Betti numbers of Hilbert schemes

Betti numbers count the dimension of certain vector spaces of differential forms of a manifold.
For a Hilbert scheme X, let $b_j(X) := \dim(H_j(X, \mathbb{Q}))$ be the *Betti numbers*, where $H_j(X, \mathbb{Q})$ denotes the j-th homology group of X with rational coefficients.

$$B(a, b; X) := \sum_{j \equiv a \pmod{b}} b_j(X)$$

We define the Hilbert schemes

$$X_1 := \mathrm{Hilb}^{n,n+1,n+2}(0), \qquad X_2 := \mathrm{Hilb}^{n,n+2}(0),$$
$$X_3 := \mathrm{Hilb}^{n,n+2}\left(\mathbb{C}^2\right)_{\mathrm{tr}}, \qquad X_4 := \widehat{M}^m(c_N),$$

where $m \in \mathbb{N}$ and c_N is some prescribed homological data.

Betti numbers of Hilbert schemes

C.—Craig—Males, 2021

Let $0 \leq a < b$ with $b \geq 2$ and

$$d(a,b) := \begin{cases} \frac{1}{b} & \text{if } b \text{ is odd,} \\ \frac{2}{b} & \text{if } a \text{ and } b \text{ are even,} \\ 0 & \text{if } a \text{ is odd and } b \text{ is even.} \end{cases}$$

Then as $n \to \infty$ we have that

$$\frac{1}{2} B(a,b;X_1) \sim B(a,b;X_2) \sim B(a,b;X_3) = \frac{d(a,b)\sqrt{3}}{4\pi^2} e^{\pi\sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right)$$

and

$$B(a,b;X_4) = \frac{d(a,b) n^{\frac{m-2}{2}}}{6^{\frac{1-m}{2}} 2\sqrt{2} c_m \pi^m} e^{\pi\sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

where $\prod_{j=1}^{m} \frac{1}{1-e^{-jz}} = \frac{1}{c_m z^m} + O(z^{-m+1})$.

A particular scheme of Göttsche

Let K be an algebraically closed field.
Let $\boldsymbol{m}$ be the maximal ideal in $K[[x, y]]$, and define

$$V_{n,K} := \mathrm{Hilb}_n\left(\mathrm{spec}\left(K[[x,y]]/\boldsymbol{m}^n\right)\right).$$

$$v(a, b; n) := \#\text{ of cells of } V_{n,K} \text{ whose dimension is congruent to } a \pmod{b}$$

C.–Craig–Males, 2021

Let $0 \leq a < b$ and $b \geq 2$. As $n \to \infty$ we have that

$$v(a, b; n) = \frac{1}{b} p(n) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

Proof for crank

Using orthogonality of roots of unity we have

$$\sum_{n\geq 0} M(a,b;n)q^n = \frac{1}{b}\sum_{n\geq 0} p(n)q^n + \frac{1}{b}\sum_{j=1}^{b-1}\zeta_b^{-aj} C\left(\zeta_b^j;q\right),$$

where

$$C(\zeta;q) := \frac{(q;q)_\infty}{F_1(\zeta;q)\,F_1(\zeta^{-1};q)},$$

with $(q;q)_\infty := \prod_{\ell=1}^{\infty}(1-q^\ell)$ and $F_1(\zeta;q) := \prod_{n=1}^{\infty}(1-\zeta q^n)$.

Proof for crank

As $z \to 0$ in D_θ, for $q = e^{-z}$ and ζ a primitive b-th root of unity (Bringmann–Craig–Males–Ono)

$$F_1\left(\zeta; e^{-z}\right) = \frac{1}{\sqrt{1-\zeta}}\, e^{-\frac{\zeta\Phi(\zeta,2,1)}{z}} \left(1 + O\left(|z|\right)\right),$$

where Φ is the *Lerch's transcendent*

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}.$$

On the major arc (Bringmann–Dousse)

$$\left(e^{-z}; e^{-z}\right)_\infty^{-1} = \sqrt{\frac{z}{2\pi}}\, e^{\frac{\pi^2}{6z}} (1 + O(|z|)),$$

while on the minor arc, for some $\mathcal{C} > 0$

$$\left|\left(e^{-z}; e^{-z}\right)_\infty^{-1}\right| \leq x^{\frac{1}{2}} e^{\frac{\pi^2}{6x} - \frac{\mathcal{C}}{x}}.$$

Proof for crank

Using the definition of $F_1(\zeta; q)$

$$\left|\operatorname{Log}\left(\frac{1}{F_1(\zeta;q)}\right)\right| = \left|\sum_{k\geq 1}\frac{\zeta^k}{k}\frac{q^k}{1-q^k}\right|$$
$$\leq \left|\frac{\zeta q}{1-q}\right| - \frac{|q|}{1-|q|} + \log\left(\frac{1}{(|q|;|q|)_\infty}\right).$$

$$\Rightarrow \qquad \left|\frac{1}{F_1(\zeta;q)}\right| \ll e^{-\frac{\mathcal{C}'}{x}}(|q|;|q|)_\infty^{-1},$$

for some $\mathcal{C}' > 0$.

Proof for crank

Since an analogous calculation holds for $F_1(\zeta^{-1}; q)$ one may conclude that

$$\left| C\left(\zeta_b^j; q\right) \right| < \left| (q;q)_\infty^{-1} \right|$$

on the minor arc.

For the major arc

$$C(\zeta; q) \ll e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{z}\right) + \operatorname{Re}\left(\frac{\zeta \Phi(\zeta,2,1)}{z}\right) + \operatorname{Re}\left(\frac{\zeta^{-1}\Phi(\zeta^{-1},2,1)}{z}\right)}.$$

Proof for crank

Therefore

$$C\left(\zeta_b^j; q\right) = o\left((q;q)_\infty^{-1}\right)$$

if and only if

$$\left(\frac{\pi^2}{3} - \varepsilon - \phi_1 - \phi_1'\right)\frac{x}{|z|^2} > \left(\phi_2 + \phi_2'\right)\frac{y}{|z|^2},$$

where $\phi_1 + i\phi_2 := \zeta_b^j \Phi(\zeta_b^j, 2, 1)$ and $\phi_1' + i\phi_2' := \zeta_b^{-j}\Phi(\zeta_b^{-j}, 2, 1)$.

Proof for crank

Note that $\phi_1 = \frac{\pi^2}{6} - \frac{\pi^2 j}{b}\left(1 - \frac{j}{b}\right) = \phi_1'$ and $\phi_2 = -\phi_2'$.
Therefore, our assumption reduces to

$$\left(\frac{2\pi^2 j}{b}\left(1 - \frac{j}{b}\right) - \varepsilon\right)\frac{x}{|z|^2} > 0,$$

which holds, since we have $b > 0$, $1 \leq j \leq b - 1$ and $x = \mathrm{Re}(z) > 0$. □

Proof for Betti numbers

Let X be a Hilbert scheme

$$G_X(T;q) := \sum_{n\geq 0} P(X;T)q^n,$$

with $P(X;T) := \sum_j b_j(X)T^j$ the *Poincaré polynomial*.

Using orthogonality of roots of unity

$$\sum_{n\geq 0} B(a,b;X)q^n = \frac{1}{b}\sum_{r=0}^{b-1} \zeta_b^{-ar} G_X\left(\zeta_b^r;q\right).$$

Proof for Betti numbers

Boccalini's thesis states that

$$G_{X_1}(\zeta; q) = \sum_{n \geq 0} P\left(X_1; \zeta\right) q^n = \frac{1+\zeta^2}{(1-\zeta^2 q)(1-\zeta^4 q^2)} F_3\left(\zeta^2; q\right)^{-1},$$

where $F_3(\zeta; q) := \prod_{n=1}^{\infty}\left(1 - \zeta^{-1}(\zeta q)^n\right)$.
We obtain

$$\begin{aligned} H_{X_1}(a, b; q) &:= \sum_{n \geq 0} B(a, b; X_1) q^n \\ &= \frac{1}{b}\left(1 + (-1)^a \delta_{2|b}\right) G_{X_1}(1; q) + \frac{1}{b} \sum_{\substack{0 < r \leq b-1 \\ r \neq \frac{b}{2}}} \zeta_b^{-ar} G_{X_1}\left(\zeta_b^r; q\right). \end{aligned}$$

Proof for Betti numbers

Since

$$G_{X_1}(1;e^{-z}) = \frac{2}{(1-e^{-z})(1-e^{-2z})}(e^{-z};e^{-z})_\infty^{-1}$$
$$= \left(\frac{1}{z^2} + \frac{3}{2z} + \frac{11}{12} + O(z)\right)(e^{-z};e^{-z})_\infty^{-1},$$

the asymptotic behaviour is essentially controlled by the Pochhammer symbol.

Using the asymptotic behaviour of $(q;q)_\infty$ we see that

$$G_{X_1}(1;e^{-z}) = \frac{1}{\sqrt{2\pi}z^{\frac{3}{2}}} e^{\frac{\pi^2}{6z}}(1+O(|z|)).$$

For $\zeta_b^r \neq 1$ it is enough to show that on the major and minor arcs,

$$G_{X_1}(\zeta_b^r;q) = o(G_{X_1}(1;q)).$$

Proof for Betti numbers

On the major arc (Bringmann–Craig–Males–Ono)

$$F_3(\zeta_b^{2r}; e^{-z})^{-1} \ll e^{\frac{\pi^2}{6z}} |z|^{-N},$$

for any $N \in \mathbb{N}$ and therefore we see that $G_{X_1}(\zeta_b^r; q) = o(G_{X_1}(1; q))$.

On the minor arc we obtain that

$$\left| F_3\left(\zeta_b^{2r}; q\right)^{-1} \right| < \left| (q; q)_\infty^{-1} \right|$$

and therefore again $G_{X_1}(\zeta_b^r; q) = o(G_{X_1}(1; q))$.

Proof for Betti numbers

Thus toward $z = 0$ on the major arc we have

$$H_{X_1}(a, b; e^{-z}) = \frac{d(a, b)}{\sqrt{2\pi} z^{\frac{3}{2}}} e^{\frac{\pi^2}{6z}} (1 + O(|z|)).$$

We are left to apply BCMO with $A = \frac{\pi^2}{6}, B = -\frac{3}{2}$, and $\alpha_0 = \frac{d(a,b)}{\sqrt{2\pi}}$ which yields that

$$B(a, b; X_1) = \frac{\sqrt{3} d(a, b)}{2\pi^2} e^{\pi\sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

from which one may also conclude asymptotic equidistribution.

Proof for Betti numbers

Similarly, it is known that

$$
\begin{aligned}
G_{X_2}(\zeta;q) &:= \frac{1+\zeta^2-\zeta^2 q}{(1-\zeta^2 q)(1-\zeta^4 q^2)} F_3\left(\zeta^2;q\right)^{-1},\\
G_{X_3}(\zeta;q) &:= \frac{1}{(1-\zeta^2 q)(1-\zeta^4 q^2)} F_3\left(\zeta^2;q\right)^{-1},\\
G_{X_4}(\zeta;q) &:= F_3\left(\zeta^2;q\right)^{-1} \prod_{j=1}^{m} \frac{1}{1-\zeta^{2j} q^j}.
\end{aligned}
$$

An analogous argument to the case of X_1 holds. □

Thank you for your attention!