Dissertation

Automorphic forms in string theory:
From Moonshine to Wall Crossing

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Prof. Dr. Anton Rebhan & Univ. Asst. Timm Wrase (PhD)
E–136 Institut für Theoretische Physik
Technische Universität Wien, AT

eingereicht an der Technischen Universität Wien
Fakultät für Physik

von
Abhiram Mamandur Kidambi (MSci, MSc)
Matrikelnummer: 01634389

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Wien, am 25.08.2020 ____________________

Timm Wrase, PhD	Prof. Dr. Gabriel Cardoso	Prof. Dr. Justin David
(TU Wien & Lehigh)	(IST Lisbon)	(IISc, Bangalore)
(Betreuer)	(Gutachter)	(Gutachter)

Automorphic Forms in String Theory: From Moonshine to Wall Crossing

Abhiram M Kidambi

Institute for Theoretical Physics, Technical University of Vienna & Stanford Institute for Theoretical Physics, Stanford University

Email address: abhiram.kidambi@tuwien.ac.at

Dedicated to the memory of M. K. Vijayaraghavan

கற்றது கை மண் அளவு கற்காதது உலக அளவு

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Abstract of this thesis

This thesis is devoted to the study of applications of automorphic forms ((mock) modular forms and (mock) Jacobi forms on $SL(2,\mathbb{Z})$ and their congruence subgroups, and Siegel modular forms on $Sp(2,\mathbb{Z})$) to moonshine phenomena, and to BPS wall crossing of dyons in $\mathcal{N}=4,\ d=4$ string theory. Moonshine phenomena are deep mathematical relations between vertex operator algebras, (mock) modular forms and sporadic groups that manifests as the encoding of the dimension of irreducible representations of a sporadic group in the coefficients of the Fourier/character expansion of a (mock) modular form. This thesis is devoted partly to understand the nature of Mathieu moonshine. In Mathieu moonshine, the elliptic genus of the $K3$ surface when expanded in characters of an $\mathcal{N}=(4,4)$ superconformal algebra has expansion coefficients that capture the dimensions of the irreducible representations of the largest Mathieu group, $\mathbf{M}_{24}$. While this phenomenon has been proven to be valid and not just a mere fluke, its exact nature and origin is still unclear. Is it a property of the Jacobi form that, up to a multiplicative factor, is the elliptic genus of $K3$, or is it a property of the moduli space of $K3$? We explore the first question by constructing and studying the elliptic genera of higher dimensional Calabi–Yau manifolds whose elliptic genus contains the very same Jacobi form as in the elliptic genus of $K3$, emphasizing on Calabi–Yau 5–folds (CY_5). The CY_5 are constructed in such a way that any connection to $\mathbf{M}_{24}$ should be apparent when expanded in superconformal characters. Extensive analysis of a large class of CY_5 rules out the possibility that $\mathbf{M}_{24}$ is an exact symmetry of the Calabi–Yau manifolds studied, thereby placing $K3$ surfaces at the center of the mathematical mystery that is Mathieu moonshine. The second part of this thesis is devoted to the exact computation of single center $\frac{1}{4}$–BPS black hole degeneracies in $\mathcal{N}=4,\ d=4$ string compactification. This theory is an ideal candidate to establish number theoretic techniques to count the BPS black hole degeneracies since it does not suffer from extensive wall crossing, nor is it overly constrained by supersymmetry. $\frac{1}{4}$–BPS states in these theories suffer from wall crossing effects i.e., their degeneracies are not the same everywhere in moduli space due to the formation/dissolution of BPS bound states. One may use number theoretic tools to remove the effects of wall crossing and compute the degeneracies for those $\frac{1}{4}$–BPS states that do not form bound states (i.e., single center states) and determine their degeneracies exactly by computing the Fourier coefficients of a particular mock Jacobi form. However, the mock Jacobi form that generates the degeneracies of single center $\frac{1}{4}$–BPS states still requires the presence of bound states of BPS instantons. These bound states are what control the growth of the black hole degeneracy in the asymptotic limit and must therefore be carefully accounted for. This is a subtle problem owing to effects of bound state metamorphosis where two or more bound states of BPS instantons get identified under the right circumstances. Therefore, one needs a consistent way in which to count their degeneracies without over–counting them. This formula is derived in this thesis and is corroborated with numerical evidence from the indexed partition function of dyonic $\frac{1}{4}$–BPS states.

List of publications

This thesis is based on the following papers/publications:

(1) *Calabi–Yau manifolds and sporadic groups* [**BCK+18**].
(2) *Dyonic black hole degeneracies in* $\mathcal{N} = 4$ *string theory from Dabholkar–Harvey degeneracies* [**CKM+19**].

The aim of this thesis is to supplement the results of the above papers with sufficient background material while also reviewing most of the results of the publications. Together, this thesis and the above papers should serve to give a self–consistent scholarship of this dissertation. The following publications, despite also being original works to which I have made significant contributions will not be expanded upon in this thesis:

(1) *On Mathieu moonshine and Gromov–Witten invariants* [**BCKS20**].
(2) *Heterotic strings on* $(K3\times T^2)/\mathbb{Z}_3$ *and their dual Calabi–Yau threefolds* [**BCK+20**].

This thesis is solely based on work that has already been published or submitted to a journal for publication. There are no current on going projects mentioned in this thesis. All the work expanded upon in this thesis are entirely my contributions. This includes computations that I have performed and sections I have (co-)authored. Instances where I may write about something that did not evolve from my contribution to a paper I co-authored have been cited.

Part 1

Introduction and Preliminaries

CHAPTER 1

Introduction and outlook

Almost every piece of pro–string theoretic literature starts with the exact same plaudits of the theory being a leading candidate to obtain a quantum formulation of gravity. Almost every piece of literature that criticizes string theory invokes the same argument i.e., there is a lack of easily falsifiable predictions. While this thesis is definitely written by a string theorist, the motivation of this thesis lies not in seeing string theory as being a theory that can work miracles in physics, but rather as a theory that acts to gracefully marry physics and mathematics. This is evident from the variety of different problems being currently investigated that require insights from both mathematics and string theory (or some limit thereof). For example, string theory (more precisely, quantum fields that arise from string theory in the limit of gravitational decoupling) has had tremendous implications for the study of 4–manifolds [**SW94a**, **SW94b**, **Wit94**, **DK90**], and in the study of the geometrization of the Langlands program [**KW07**, **Fre07**]. Both these questions are cutting edge problems in geometry and the theory of automorphic forms.

In this thesis, we shall mostly be interested in the role of number theory and geometry in string theory, and vice versa. More concretely, we will be interested in two different problems in which geometry, automorphic forms and string theory play a vital role. Automorphic forms are highly symmetric functions with constrained transformation properties and these functions are useful in determining symmetric properties of theories with a large number of parameters or high degeneracy, for example in partition functions of highly degenerate black holes, computation of invariants of manifolds whose moduli space has large dimensionality. While there are many different applications of automorphic forms even within the context of string theory (see concluding remarks of Chapter 2), we shall primarily be focused on only two applications.

1.1. Moonshine

The first of these applications, moonshine, originated more as a mathematical enigma in the late 1970's. The oddity stemmed from the fact that the Fourier expansion coefficients of the modular function $J(\tau)$ can be written as linear combinations of the dimensions of the irreducible representations of the Monster sporadic group, $\mathbb{M}$. This initial observation eventually grew from an interesting mathematical observation into a heavy mathematical theory which related sporadic groups, lattices, modular forms, conformal field theories and even invoking the Goddard–Thorn no–ghost theorem for bosonic string theory in its proof [**GT72**]. This entire enterprise culminated in the formulation of a mathematical theory known as *monstrous moonshine* since it is a 'moonshine' phenomenon for the Monster group.

More surprises from a string theoretic direction arose when the dimensions of irreducible representations of another sporadic group, $\mathbf{M}_{24}$, were discovered to be recoverable from expanding the elliptic genus of $K3$ in terms of characters of an $\mathcal{N} = (4, 4)$

superconformal field theory. This is known as *Mathieu moonshine*. While this, too, has been proven, the exact nature and origin of this moonshine is still unresolved. This will be the key focus of the first half of the research part of this thesis, as presented in Chapter 5 and Chapter 6. The nature of the question that we shall ask in Chapter 6 is if whether the Mathieu moonshine is a phenomenon associated to $K3$ surfaces, or whether it is a property of a Jacobi form that defines the elliptic genus of $K3$. We will go over the systematic construction of higher dimensional Calabi–Yau manifolds and their elliptic genera and search for moonshine phenomena by twining their elliptic genera. The logic here is that these higher dimensional Calabi–Yau manifolds have elliptic genera that are proportional to the elliptic genus of $K3$, and are hence prone to the same moonshine phenomena if Mathieu moonshine is a property of the Jacobi form that appears in the $K3$ elliptic genus. Despite a significant amount of analysis, the conclusion that we shall arrive at is that Mathieu moonshine is very likely a special property of $K3$. The results of this project can be found in [**BCK+18**]. This naturally leaves us with more questions, the first of which is

What is the role of $K3$ in Mathieu moonshine?

The most promising approach to answer this question is a technique known as symmetry surfing in which the idea is to scan the moduli space of $K3$ surfaces for all distinct geometric symmetries of the $K3$ at special points and this group be the Mathieu group, $\mathbf{M}_{24}$. However, this approach (as of now) falls short of generating the full $\mathbf{M}_{24}$ group. There are of course broader questions that one can ask.

Can the linear combinations of dimensions of the irreducible representations of any other sporadic group be equal to expansion coefficients of some modular object?

The answer to this question is 'yes'. Many other moonshine phenomena have been found to exist, but their exact origin is unclear. More so, proposed physical interpretations for various moonshines seem to be philosophically different.

Can moonshine phenomena all be described in the same way i.e., by the same logic underlying physical theories?

While this is a much more difficult question to answer at this point in time, progress has been reported in this direction in [**BHL+20**].

Are these sporadic groups discrete symmetry groups of string theoretic object?

While this question, too, is not exactly easy to answer, there is an element of weight to this question. In particular, it is possible that these sporadic groups are in some way related to the symmetries of the BPS states in the string theory. These topics and questions will be addressed, with key references, in Chapter 5 and Chapter 6.

1.2. Counting BPS states in string theory

The second of the two applications of automorphic forms in this thesis will be in the study of degeneracies of supersymmetric black holes. The physical motivation of having to compute black hole degeneracies is to answer the question

What are the microscopic degrees of freedom of a black hole?

Black holes in some sense are both the most and least complicated objects in a gravitational theory that permits black holes. Most complicated since they are highly exotic objects with high degeneracy, and least complicated in the sense that their dynamics can be almost exactly computed. To answer the above question requires a robust theory of quantum gravity and string theory has been able to do so mostly for a class of supersymmetric black holes known as extremal or BPS black holes. These black holes do not radiate and therefore understanding their microscopic structure is much easier than in the non–BPS case. However, computation of the microscopic degrees of freedom of a black hole is not a simple task due to its high degeneracy. The second advantage of BPS black holes is that the spectrum of BPS states does not vary with string coupling. This means that a bound state of branes and strings, which can be described by a conformal field theory at low string coupling, has the same BPS spectrum even if the string coupling is tuned high enough that the system collapses into a black hole. Therefore, for these black holes, computing their BPS partition function can be done in the weakly coupled conformal field theory limit. There is however a subtlety here in that while the BPS spectrum is invariant under changes in string coupling, it is not necessarily invariant under changes to the moduli. Thus the question here would be to determine a technique for counting BPS states that would in some sense be immune to changes in the moduli of the theory and/or derive a function that would keep track of BPS degeneracies at all points in moduli space. Such functions are often automorphic forms, and their Fourier expansion coefficients which change with moduli encode BPS black hole degeneracies. Given a function that generates BPS degeneracies, extracting them precisely with as minimal amount of input data is necessary owing to the computational complexity of the problem. By analyzing subtleties in the wall crossing phenomena for $\frac{1}{4}$–BPS states in $\mathcal{N}=4,\ d=4$ string theory, we are able to derive an exact formula that keeps track of this minimal amount of information (i.e., the so called polar states) that is required to reconstruct the full black hole entropy. Understanding the above statements and the results of [**CKM+19**] will the focus of Chapter 7 and Chapter 8.

1.3. How to read this thesis

This thesis comprises of two different scientific problems, the first one related to Mathieu moonshine, and the second one related to BPS black hole degeneracies in $\mathcal{N}=4,\ d=4$ string theory. The first part of the thesis comprises of reviewing pre–requisite material on automorphic forms (Chapter 2), Calabi–Yau geometry (Chapter 3) and finite group theory (Chapter 4).

The next part of the thesis is devoted to moonshine phenomena for which all the introductory chapters are invoked as pre–requisites. We provide an introduction to moonshine phenomena (focusing only on the Monstrous and Mathieu moonshines) (Chapter 5). Our analysis of studying higher dimensional Calabi–Yau manifolds and their elliptic genera and relations to sporadic groups follows right after (Chapter 6). The relevant notation for superconformal characters can be found in Appendix A, while the character tables for the two important Mathieu groups, $\mathbf{M}_{24}$ and $\mathbf{M}_{12}$, can be found in Appendix B.

The final part of the thesis is devoted to the study of automorphic forms and black holes. Chapter 2 and Chapter 3 form the pre–requisites for this part. The first chapter in this part is devoted to developing the necessary pre–requisites to understand the number theoretic aspects of BPS black holes in $\mathcal{N}=4,\ d=4$ theories (Chapter 7) and the final chapter of this thesis will be devoted to deriving an exact formula for the degeneracies of polar states of certain mock Jacobi forms which determine the entropy of single center $\frac{1}{4}$–BPS black holes (Chapter 8). Numerical data to provide evidence for our analysis in this chapter can be found in Appendix C.

CHAPTER 2

Automorphic forms on $SL(2,\mathbb{Z})$, $Sp(2,\mathbb{Z})$, and their generalizations

Overview of this chapter

In this chapter, we introduce the concepts pertaining to automorphic forms on $SL(2,\mathbb{Z})$, $Sp(2,\mathbb{Z})$ (modular forms, Jacobi forms and Siegel modular forms) that are required in the exploration of this thesis. A note to the reader: While dealing with modular forms, Jacobi forms and Siegel modular forms, often in the mathematical literature one finds that the exponent of the function is placed before invoking the modular/elliptic parameters. An example of this is $\phi^2_{k,m}(\tau, z)$. An equivalent approach that is used in this thesis is to place the exponent after the parameter as $\phi_{k,m}(\tau, z)^2$. Both these notations mean the same thing, mathematically speaking.

2.1. Modular forms

In this section we provide an overview of the concepts in number theory required for a reader to understand this thesis. We begin with the definition of the upper half plane (UHP), $\mathbb{H}$.

Definition 2.1.1 (UHP). *The UHP is the set of all complex numbers whose imaginary part is greater than zero i.e.,* $\mathbb{H} = \left\{\tau \in \mathbb{C} \;\middle|\; \mathrm{Im}\,(\tau) > 0\right\}$.

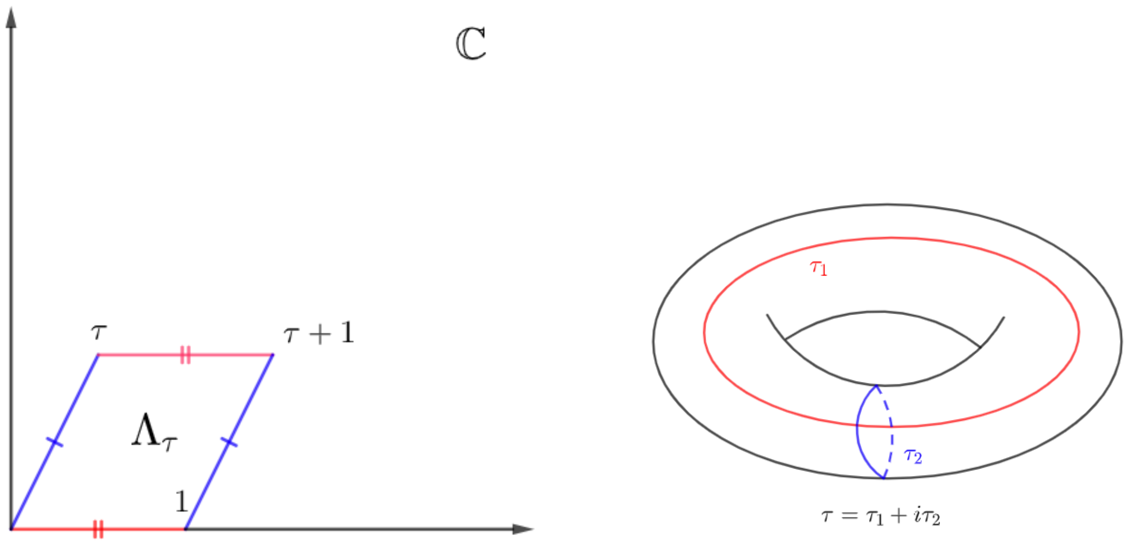
The UHP admits an action of the $SL(2,\mathbb{R})$ group via fractional Möbius transformations of the Riemann sphere as follows

$$SL(2,\mathbb{R}) : \tau \to \frac{a\tau + b}{c\tau + d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,. \tag{2.1}$$

Of importance here is the action of the discrete subgroup $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$. The group $SL(2,\mathbb{Z})$ is the modular group of the torus in the sense that each τ defines a complex torus with $\tau = \tau_1 + i\tau_2$.

The moduli space of T^2 tori, which are surfaces of Euler characteristic $\chi = 0$, is given by $\mathbb{H}/SL(2,\mathbb{Z})$. $SL(2,\mathbb{Z})$[1] has a well defined action on certain functions on the UHP. These functions are modular functions/forms and will the focus of this thesis.

[1] $SL(2,\mathbb{Z}) \cong SO(2,1)$.



(A) Lattice of T^2, Λ_τ (B) $T^2 \cong \mathbb{C}/\Lambda_\tau$

FIGURE 1. Depiction of a torus and its lattice construction. For each value of τ in the UHP, there is a unique torus associated to it, thereby defining the moduli space of 2-tori.

DEFINITION 2.1.2 (Modular forms). *A modular form of weight k is a holomorphic function $f(\tau): \mathbb{H} \to \mathbb{C}$ defined on the UHP which transforms as follows under the action of $SL(2,\mathbb{Z})$:*

$$f(\tau) \to f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{2.2}$$

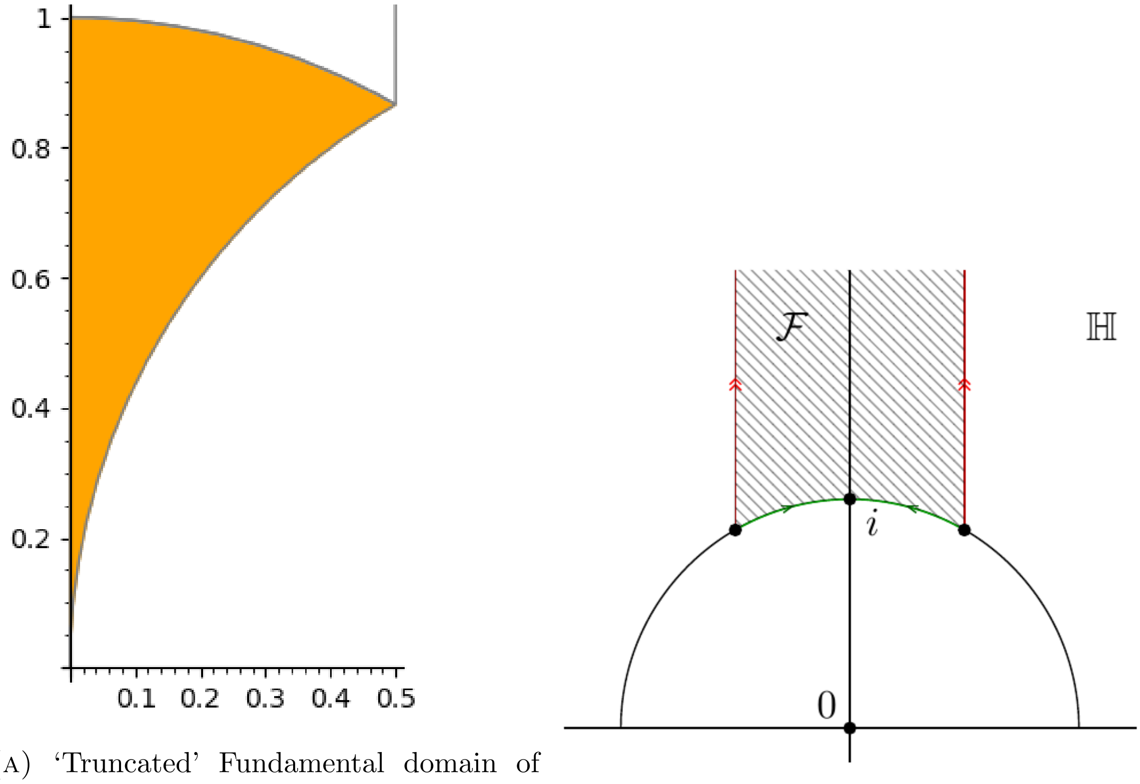
A modular form is holomorphic in the limit that $\tau \to i\infty$. Since a modular form transforms under the action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ i.e., $\tau \to \tau+1$, it is a periodic function and therefore admits a Fourier expansion on the open unit disc $D = \{q \in \mathbb{C}, |q| < 1\}$, $q = e^{2\pi i\tau}$, as follows:

$$f(\tau) = \sum_n a_n q^n, \quad q = e^{2\pi i \tau}. \tag{2.3}$$

REMARK 2.1.1. *Equivalently, modular forms are also defined as a function from the set of complex lattices to $\mathbb{C}$ satisfying the following properties:*

(1) A complex lattice is a lattice that is generates by a pair (A,τ) as $\Lambda = \mathbb{Z}A + \mathbb{Z}\tau$. A modular form, $f(\Lambda)$ is a function that is analytic on this lattice.
(2) The weight k of the modular form is the scaling factor associated to the analytic function as $f(\Lambda) \to f(a\Lambda) = a^{-k} f(\Lambda)$.
(3) The norm of the function $f(\Lambda)$ is bounded from above if and only if the norm of the smallest element of the lattice Λ is non-zero.

This definition of a modular form helps illuminate the relation between modular functions and lattices that is manifest in the study of moonshine.



(A) 'Truncated' Fundamental domain of $SL(2,\mathbb{Z})$.

(B) Fundamental domain, $\mathcal{F}$, of $SL(2,\mathbb{Z})$.

FIGURE 2. The fundamental domain is obtained by considering the mirror image of the truncated domain about the y–axis which represents $\mathrm{Re}(\tau) = 0$.

DEFINITION 2.1.3 (Modular functions). *Modular functions are meromorphic generalizations of modular forms of weight zero. These functions are still periodic and admit a Fourier expansion on the open unit disk D of the form*

$$f(\tau) = \sum_{n=-|m|} a_n q^n.$$

The fact that there are only a finite number of terms with negative powers of q means that the function is bounded from below in its q–expansion and is therefore meromorphic at $\tau = i\infty$.

Modular forms on $SL(2,\mathbb{Z})$ have the property that their behaviour at $\tau = 0$ is mapped to their behaviour at $\tau = i\infty$. This identification allows the interpretation/definition of a modular form as a function defined on the compact fundamental domain $SL(2,\mathbb{Z})\backslash\mathbb{H}$ to the Riemann sphere $\mathbb{C}\cup\{\infty\}$. Shown in Figure 2 is a '*truncated*' fundamental domain for $SL(2,\mathbb{Z})$. The fundamental domain is obtained by considering the mirror image of the truncated domain about the y–axis i.e., $\mathrm{Re}(\tau) = 0$.[2] The modular forms on $SL(2,\mathbb{Z})$ are even weight forms that are generated by the *Eisenstein series.*[3]

[2]This approach to constructing fundamental domains using Farey arcs was studied in [**Kul91**].

[3]Eisenstein series are comprehensive area of study in analytic number theory and we point the reader to [**FGKP18**] for a more detailed study.

The Eisenstein series themselves are modular forms on $SL(2,\mathbb{Z})$. The free action of the two Eisenstein series on $SL(2,\mathbb{Z})$ as defined below generates a ring of modular forms on $SL(2,\mathbb{Z})$:

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} = 1 + 240\, q + 2160\, q^2 + \cdots . \tag{2.4a}$$

$$E_6(\tau) = 1 + 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} = 1 + 504\, q + 16632\, q^2 + \cdots . \tag{2.4b}$$

Other modular forms may be constructed using the Eisenstein series. The space of all holomorphic modular forms of weight k is denoted by M_k. Any modular form of even weight $k > 2$ is expressible as a linear combination of products of Eisenstein series

$$f_k(\tau) = \sum_{(\alpha,\beta)\big|k=4\alpha+6\beta} C_{(\alpha,\beta)}\; E_4(\tau)^{\alpha}E_6(\tau)^{\beta}, \tag{2.5}$$

where the $C_{(\alpha,\beta)}$'s are real coefficients. For example, the *Ramanujan discriminant*, $\Delta(\tau)$, can be defined as

$$\Delta(\tau) := \frac{E_4(\tau)^3 + E_6(\tau)^2}{1728} = \eta(\tau)^{24}. \tag{2.6}$$

Here, $\eta(\tau)$ is the Dedekind-eta function and is defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,, \tag{2.7a}$$

$$\frac{1}{\eta(\tau)^{24}} = q^{-1}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{24}}\,, \tag{2.7b}$$

and will play an important role in this thesis. It is therefore useful to state a few remarks on the Dedekind-eta function. $\eta(\tau)$ appears in the partition functions of bosons. Mathematically, the function $\dfrac{1}{\eta(\tau)^{24}}$ appears in the theory of partitions as studied by Ramanujan and Hardy as the index partition of 24 colours [**HR18a**]. In physics, the Dedekind-eta appears in the partition function of various conformal field theories. The CFT which will be of interest to us is the CFT of 24 free bosons compactified on a torus T^2 which has precisely (2.7b) as its partition function and is the partition function of $\frac{1}{2}$−BPS states in the four dimensional $\mathcal{N} = 4$ string compactification [**DH89**]. There also exists an Eisenstein series of weight 2,

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} = q - 24\, q^2 + 252\, q^3 + \cdots . \tag{2.8}$$

However, this is a quasi–modular form since it is not modular but can be completed in a non-holomorphic manner to be modular. This non–holomorphic completion is as follows

$$\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}\,(\tau)}\,. \tag{2.9}$$

The above function (2.9) transforms as a weight 2 form. The physical importance of such forms will become apparent in Chapter 7 and Chapter 8. These forms are used to define the Ramanujan-Serre derivative which transform a weight k modular form into a weight $k+2$ modular form

$$\begin{aligned} \partial_k^{RS} &: M_k \to M_{k+2}\,, \\ \partial_k^{RS} &:= \frac{1}{2\pi i} f'(\tau) - \frac{k}{12} E_2(\tau) f(\tau)\,. \end{aligned} \tag{2.10}$$

Having introduced the concept of a modular form, we now turn to classification of modular forms.

2.1.1. Classification of modular forms. Fourier expansions of modular forms contain a substantial amount of information about them. Therefore, we may use the properties of the Fourier expansion to classify (in a certain sense) modular forms. We recall the Fourier expansion of a modular form $f(\tau)$ stated in (2.3). The set of all modular forms may be holomorphic (in M_k, as encountered before), cuspidal or weakly holomorphic.

Cusp form: A cusp form is a modular form such that $a_0 = 0$. Since the value of the modular form at zero is related by modular transforms to its value at $\tau = i\infty$, the cusp form vanishes at complex infinity. The set of cusp forms of weight k are denoted S_k henceforth.

Weakly holomorphic modular forms (WHMF's): If $a_p = 0\ \forall p < -N$, then the modular form grows as $O(q^N)$ at $\tau = i\infty$ instead of $O(1)$. Such modular forms are known as weakly holomorphic modular forms and are denoted by $M_k^!$.

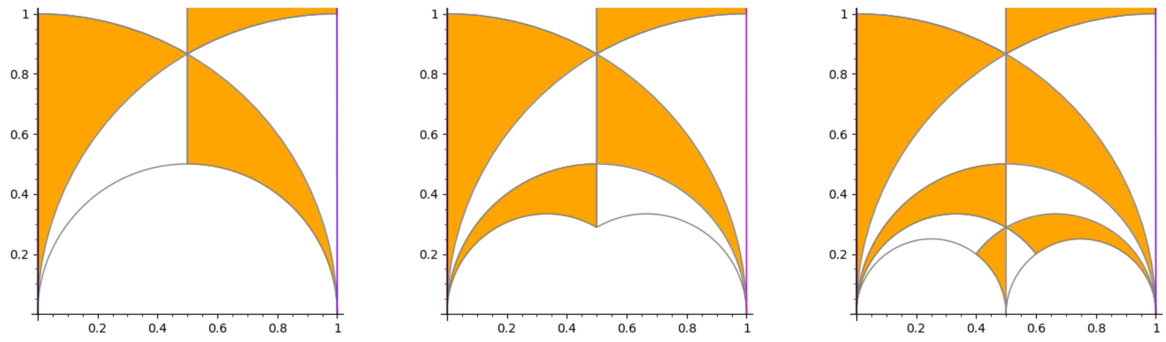
The growth of Fourier coefficients of the different kinds of modular forms are listed in the Table 1 below.

Growth of Fourier coefficients of modular forms		
Modular form	Type	Growth of a_n as $n \to \infty$
$f \in S_k$	Cusp form	$a_n = O(n^{k/2})$
$f \in M_k$	Holomorphic modular form	$a_n = O(n^{k-1})$
$f \in M_k^!$	Weakly Homomorphic modular form	$a_n = O(e^{\sqrt{n}})$

TABLE 1. Growth of Fourier coefficients of modular forms.

REMARK 2.1.2. *Since modular forms represent partition functions in physics, the different kinds of modular forms represent thermodynamic systems with different kinds of entropy scaling. For example, we shall focus on the case where the modular form is weakly holomorphic in which case the growth of degeneracies is exponential, as is required in a Boltzmannian system.*

2.1.2. Congruence subgroups (CSG's) of $SL(2,\mathbb{Z})$. We take a digression into congruence subgroups of $SL(2,\mathbb{Z})$. CSG's arise in the modular description when one takes a quotient or an orbifold of a theory. They make appearances in the constructions of Hauptmoduln in Chapter 5, the construction of twisted-twined elliptic genera in Chapter 5 and the CHL orbifolds of black hole partition functions in Chapter 7. The first CSG that we list here is the principle congruence subgroup $\Gamma(N) \subset SL(2,\mathbb{Z})$. This principal CSG is a normal subgroup of $SL(2,\mathbb{Z})$ since it is the kernel of the group



(A) $\Gamma_0(2)$ (B) $\Gamma_0(3)$ (C) $\Gamma_0(5)$

FIGURE 3. Truncated fundamental domains for certain CSG's of $SL(2,\mathbb{Z})$. As before in 2, the fundamental domain in each of the above cases is the mirror reflection about the y–axis.

homomorphism from $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/N\mathbb{Z})$. In other (lack of) words,

$$\Gamma(N) \subset SL(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \tag{2.11}$$

The CSG's that we will encounter in this thesis are

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| c \equiv 0 \mod N \right\}, \tag{2.12a}$$

$$\Gamma^0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| b \equiv 0 \mod N \right\}, \tag{2.12b}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \middle| c \equiv 0 \mod N,\ a \equiv 1 \mod N \right\}. \tag{2.12c}$$

An important property of modular (sub)groups that will feature in this thesis is the *genus–0* property. A modular (sub)group is genus–0 if its fundamental domain is topologically equivalent to a Riemann sphere. Many but not all of these modular groups withhold the same genus-0 property as $SL(2,\mathbb{Z})$ [**CLY04**].[4] Congruence subgroups, too, have modular forms/functions defined on them in the same manner as for $SL(2,\mathbb{Z})$. These are still periodic in the sense of $\tau \to \tau + b$ for a fractional transformation $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, $\tau \to \tau + Nb$ for a fractional transformation $\begin{pmatrix} 1 & Nb \\ 0 & 1 \end{pmatrix} \in \Gamma^0(N)$. Therefore, they admit Fourier expansions. The modular forms on $\Gamma \subset SL(2,\mathbb{Z})$ may also be defined as a mapping from an appropriate fundamental domain (see Figure 3) to the Riemann sphere. Modular forms on these arithmetic subgroups admit Eisenstein series and theta functions. We shall discuss theta series and functions in Section 2.6 mostly for $SL(2,\mathbb{Z})$, but the other topics pertaining to arithmetic subgroups are beyond the scope of pedagogy for this thesis and we refer the curious reader to [**Els85**] for more details.

[4]As we shall later see, this is precisely the reason why the overarching use of genus–0 properties for moonshine phenomena on arithmetic subgroups has been replaced with Rademacher summability.

2.1.3. Atkin–Lehner involution. For modular forms on $SL(2,\mathbb{Z})$, we have seen that their behaviour at $\tau = 0$ is related to their behaviour at $\tau = i\infty$. With modular forms on $\Gamma_0(N)$, there is a subtlety. While the periodicity property can be maintained, there is however no element in $\Gamma_0(N)$ analogous to the transformation $S : \tau \to \dfrac{-1}{\tau}$ in $SL(2,\mathbb{Z})$. This involution is brought forward by the action of a Fricke involution of the form $S' : \tau \to \dfrac{-1}{N\tau}$, which can be generalized to Atkin–Lehner involutions[**PV15**].[5] The physical importance of such transformations is the S–duality. To account for such identifications, one uses the following technique. Let N be the level of the CSG. We consider all Hall divisors of N which are those divisors x such that N/x and x are coprime. For each Hall divisor, we define

$$W_x = \begin{pmatrix} ax & b \\ cN & dx \end{pmatrix}, \ \det W_x = x\,.$$

These divisor matrices act as normalizers for any element in $\Gamma_0(N)$ and can be chosen in a way in which they act as an involution i.e., $W_x^2 = \mathbb{I}_2$. The group $\Gamma_0(N)$ together with the involution W_x comprise the group $\Gamma_0(N)^+$ which affords correct cusp identification properties.

2.2. Jacobi forms

Another class of automorphic forms are Jacobi forms. These forms were first studied in detail in [**EZ85**] and we refer the reader to this piece of literature for more details as only the rudiments are invoked in this thesis.

Definition 2.2.1 (Jacobi form). *A Jacobi form (JF) on $SL(2,\mathbb{Z})$ of weight k and index m is a holomorphic function $f_{k,m} = \phi(\tau,z) : \mathbb{H}\times\mathbb{C} \to \mathbb{C}$ that has modular and elliptic properties. The variable τ is the modular variable, and the variable z is the elliptic variable. These properties are defined as follows:*

Modularity:

$$\phi(\tau,z) \to \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi imcz^2}{c\tau+d}}\phi(\tau,z)\ \forall\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,.$$

Ellipticity:

$$\phi(\tau,z) \to \phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi im(\lambda^2\tau+2\lambda z)}\phi(\tau,z),\ \forall\ \lambda,\mu\in\mathbb{Z}\,.$$

The index of a Jacobi form on $SL(2,\mathbb{Z})$ is always a positive integer, while the weight is an integer.

From the above definition, it is clear that the modular and elliptic properties of a JF imply periodicity in both the elliptic and modular variables. Therefore, Jacobi forms admit a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{n,\ell} c(n,\ell)q^n y^\ell, \quad y = e^{2\pi iz}. \tag{2.13}$$

[5]It is this Fricke involution that plays the role of S-duality in orbifolded models.

The periodicity in the elliptic variable ensures that the Fourier coefficient $c(n,\ell) \equiv C(\Delta = 4mn - \ell^2, \ell)$[6] is fixed in terms of $\ell \mod 2m$ [EZ85, DMZ12].

2.2.1. Classification of Jacobi forms. Similar to the case of the theory of modular forms as studied in Section 2.1.1, Jacobi forms, too, can be classified in terms of their Fourier expansion coefficients.

Holomorphic Jacobi form: A holomorphic JF (HJF) of weight k and index m is a JF with $C(\Delta < 0, \ell) = 0$. The set of HJF's of weight k, index m is denoted by $J_{k,m}$.

Jacobi cusp form: A Jacobi cusp form (JCF) of weight k and index m is a JF with $C(\Delta \leq 0, \ell) = 0$. The set of JCF's of weight k, index m is denoted by $J^0_{k,m}$.

Weak Jacobi form: A weak Jacobi form (WJF) of weight k and index m is a JF with $C(\Delta = 4mn - \ell^2, \ell) = c(n,\ell) = 0$ if $n < 0$. The set of WJF's of weight k, index m is denoted by $\tilde{J}_{k,m}$.

Weakly holomorphic Jacobi form: A weakly holomorphic Jacobi form (WHJF) of weight k and index m is a JF with $C(\Delta = 4mn - \ell^2, \ell) = c(n,\ell) = 0$ if $n < n_0,\ n_0 < 0$. The set of WHJF's of weight k, index m is denoted by $\tilde{J}^{!}_{k,m}$.

2.2.2. Theta expansion of Jacobi forms. A crucial property of Jacobi forms is their representation in terms of a theta expansion. Let $\phi(\tau,z)$ be a Jacobi form. The Fourier expansion of the Jacobi form with respect to z can be expressed as

$$\phi(\tau,z) = \sum_{\ell\in\mathbb{Z}} q^{\frac{\ell^2}{4m}} h_\ell(\tau) y^\ell, \tag{2.14}$$

where $h_\ell(\tau) = \sum_{\Delta} C(\Delta,\ell) q^{\Delta/4m}$, $\ell \in \mathbb{Z}/2m\mathbb{Z}$ and $h_\ell(\tau)$ is itself periodic in terms of $\ell \equiv 2mn$. This implies that (2.14) can be expressed as

$$\phi(\tau,z) = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\, \underbrace{q^{(\ell+2mn)^2/4m} y^{\ell+2mn}}_{\vartheta_{m,\ell}(\tau,z)}, \tag{2.15}$$

where $\vartheta_{m,\ell}(\tau,z)$ is the standard theta function with index m and weight $1/2$.[7] The above equation (2.15) is the *theta decomposition* of a Jacobi form. Naturally, if $\vartheta_{m,\ell}(\tau,z)$ is a Jacobi form of weight $1/2$, then $h_\ell(\tau)$ is a Jacobi form of weight $k-1/2$. The nature of $h_\ell(\tau)$ i.e., whether it is weakly holomorphic, holomorphic or cuspidal is fixed by the nature of the Jacobi form being weak, holomorphic or cuspidal, respectively. However it is important to state that $h_\ell(\tau)$ transforms not as a scalar function on $SL(2,\mathbb{Z})$ but rather as a vector–valued modular form $h = (h_1, h_2, \cdots, h_{2m})$ of weight $k - 1/2$ on $SL(2,\mathbb{Z})$. Using the expression (2.15), we can define a differential operator that acts on Jacobi forms that preserves their ellipticity properties. This operator is a heat kernel operator of index m and is expressed as follows

$$\mathcal{L}_m = \frac{4m}{2\pi i}\frac{\partial}{\partial\tau} - \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}\,. \tag{2.16}$$

[6]The $\Delta = 4mn - \ell^2$ here is known as the discriminant. It is not to be confused with the Ramanujan discriminant $\Delta(\tau) = \eta(\tau)^{24}$.

[7]This is a Jacobi form on a subgroup of $SL(2,\mathbb{Z})$.

The action of this heat kernel operator is $\mathcal{L}_m : \tilde{J}_{k,m} \to \tilde{J}_{k,m}$ and it acts on a Jacobi form in the following way:

$$\mathcal{L}_m : \phi(\tau,z) = \sum_{n,\ell} c(n,\ell) q^n y^\ell \mapsto \phi'(\tau,z) = \sum_{n,\ell} (4mn-\ell^2) c(n,\ell) q^n y^\ell \,. \tag{2.17}$$

Since the Fourier coefficients for both $\phi(\tau,z)$, $\phi'(\tau,z)$ are the same, their elliptic transformation properties are preserved under this mapping. However, since

$$\mathcal{L}_m \left(\sum h_\ell \, \vartheta_{m,\ell}\right) = 4m \sum h'_\ell \, \vartheta_{m,l} \ , \tag{2.18}$$

we may now construct another operator which is the modified heat kernel operator

(2.19)
$$\mathcal{L}_{k,m} = \mathcal{L}_m - \frac{m\left(k-\frac{1}{2}\right)}{3} E_2(\tau) : \sum_\ell h_\ell(\tau)\vartheta_{m,\ell}(\tau,z) \mapsto 4m \sum_\ell \partial^{RS}_{k-\frac{1}{2}} h_\ell(\tau)\vartheta_{m,l}(\tau,z),$$

where ∂^{RS}_* is the Ramanujan-Serre derivative as defined in (2.10). The action of this modified heat kernel operator (2.19) is to send $J_{k,m}/\tilde{J}_{k,m}/J^!_{k,m} \to J_{k+2,m}/\tilde{J}_{k+2,m}/J^!_{k+2,m}$. The theta expansion of a Jacobi form will be of use in constructing the Rademacher expansion which will be discussed in Section 2.7.

2.2.3. Examples of Jacobi forms. As for the case of modular forms where the two Eisenstein series (2.4) generated the ring of modular forms $M_k = \langle E_4(\tau), E_6(\tau)\rangle$ on $SL(2,\mathbb{Z})$, we may analogously construct a bigraded ring of Jacobi forms on $SL(2,\mathbb{Z})$. The starting point for this is the construction of all index 1 Jacobi forms. Since the product of a modular form and a Jacobi form still has elliptic properties, non–trivial weight Jacobi forms can also generated by the action of the two Eisenstein series. The full construction of index 1 Jacobi forms is given in [**DMZ12**]. There are four Jacobi form's of the form $\phi_{k,1}(\tau,z)$, $k=-2,0,10,12$ which are defined as follows:

$$\phi_{-2,1}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6} = \frac{\phi_{10,1}(\tau,z)}{\Delta(\tau)} \,. \tag{2.20a}$$

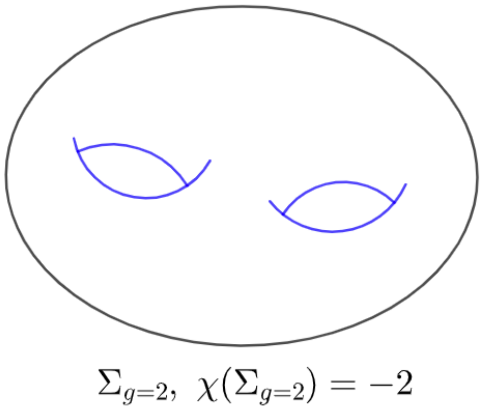
$$\phi_{0,1}(\tau,z) = 4\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right) \,. \tag{2.20b}$$

$$\phi_{10,1}(\tau,z) = \eta(\tau)^{18}\vartheta_1(\tau,z)^2 \,. \tag{2.20c}$$

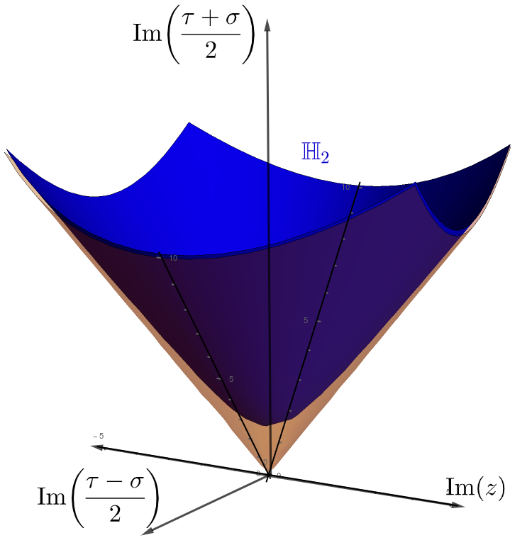
$$\phi_{12,1}(\tau,z) = \Delta(\tau)\phi_{0,1}(\tau,z) \,. \tag{2.20d}$$

Here $\vartheta_i(\tau,z)$ are the standard Jacobi theta functions defined in (2.44). Since the Eisenstein series (2.4) and the Jacobi forms (2.20) are all even weight functions, they can only generate a ring of even weight Jacobi forms.[8] Of the Jacobi forms defined in (2.20), $\phi_{10,1}(\tau,z)$ and $\phi_{12,1}(\tau,z)$ are cuspidal Jacobi forms while $\phi_{0,1}(\tau,z)$ and $\phi_{-2,1}(\tau,z)$ freely generate the ring of weak Jacobi forms.

[8] Odd weight Jacobi forms cannot have index 1 but can be constructed using isomorphisms between (weak) Jacobi forms of weight k and index 2 and (holomorphic) cuspidal modular forms of weight $k+1$ [**DMZ12**].



(A) Depiction of a genus-2 surface on which a Siegel modular form is defined.



(B) $\mathbb{H}_2$ lies in the positive light cone of $\mathbb{R}^{2,1}$ whose conformal group is $SO(3,2) \cong Sp(2,\mathbb{Z})$.

Theorem 2.2.1 (Eichler–Zagier–Feingold–Frenkel). *The bigraded ring of all weak Jacobi forms of even weight k and positive index m is a polynomial algebra in $E_4(\tau), E_6(\tau)$ and is generated by $\phi_{0,1}(\tau,z)$ and $\phi_{-2,1}(\tau,z)$* [**EZ85, FF83**]. *In other words,*

$$\tilde{J}_{k,m} = \langle E_4(\tau), E_6(\tau), \phi_{0,1}(\tau,z), \phi_{-2,1}(\tau,z)\rangle\,, \tag{2.21}$$

where the $k \in 2\mathbb{Z}$.

We do not prove this theorem here and we leave it to the reader to consult [**EZ85**] for details of the proof.

2.3. Siegel modular forms

We now discuss the theory of Siegel modular forms. While we have thus far encountered functions defined on T^2 with the modular group $SL(2,\mathbb{Z})$, there are natural extensions to this. In generalizing the space of surfaces of Euler characteristics $\chi = 0$ to surfaces of $\chi = -2$, we end up genus 2 surfaces for which the modular group extends as

$$\begin{aligned} T^2 \cong \Sigma_{g=1} &\to \Sigma_{g=2}\,, \\ SO(2,1) \cong SL(2,\mathbb{Z}) &\to SO(3,2) \cong Sp(2,\mathbb{Z})\,. \end{aligned} \tag{2.22}$$

DEFINITION 2.3.1 (Siegel Upper Half Plane, $\mathbb{H}_2$). *The Siegel upper half plane (SUHP), $\mathbb{H}_2$, is the genus two generalization of the UHP which is defined to be the set of 2×2 symmetric matrices with complex entries*

$$\mathbb{H}_2 = \left\{ \Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in Mat_{2\times 2}(\mathbb{C}) \,\middle|\, \mathrm{Im}(\tau) > 0,\ \mathrm{Im}(\sigma) > 0,\ \det(\mathrm{Im}(\Omega)) > 0 \right\}. \tag{2.23}$$

DEFINITION 2.3.2 ($Sp(2,\mathbb{Z})$). *$Sp(2,\mathbb{Z})$ is defined as the group of 4×4 matrices g that preserve the symplectic form i.e., $gJg^T = J$, where $J = \begin{pmatrix} 0 & -\mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}$ is the symplectic form. $Sp(2,\mathbb{Z})$ is the group g of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with entries 2×2 matrices that admit a natural action on $\mathbb{H}_2$ as*

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1},\ \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Z}). \tag{2.24}$$

REMARK 2.3.1. *Analogous to how τ defines the period of the 2-torus, Ω can be interpreted as the period matrix of a genus 2 Riemann surface.*

DEFINITION 2.3.3 (Siegel Modular form). *A Siegel modular form (SMF) of weight k is a holomorphic function $\Phi : \mathbb{H}_2 \to \mathbb{C}$ that transforms under $Sp(2,\mathbb{Z})$ such that*

$$\Phi(\Omega) \to \Phi\left((A\Omega + B)(C\Omega + D)^{-1}\right) = \det(C\Omega + D)^k \Phi(\Omega),\ \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Z}). \tag{2.25}$$

A SMF admits a Fourier expansion of the form

$$\Phi(\Omega) = \sum_{\substack{m,n,\ell \in \mathbb{Z} \\ 4mn - \ell^2 > 0}} c(m,n,\ell) q^n p^m y^\ell,\ p = e^{2\pi i \sigma}, \tag{2.26}$$

where Ω is a short hand for (τ, σ, z). In the above equation (2.26), the sum can be re-written by expressing $q^n y^\ell$ terms as a Jacobi form (since Jacobi forms have the expansion with $q^n y^\ell$ as seen in (2.13)). This gives us the Fourier-Jacobi expansion of a SMF as

$$\Phi_k(\Omega) = \sum_{\substack{m,n,\ell \in \mathbb{Z} \\ 4mn - \ell^2 > 0}} \underbrace{c(m,n,\ell) q^n y^\ell}_{\phi_{k,m}(\tau,z)} p^m = \sum_{m=0}^{\infty} \phi_{k,m}(\tau, z) p^m, \tag{2.27}$$

where $\phi_{k,m}(\tau, z)$ is a Jacobi form of weight k and index m. The most prominent example of the SMF that we will encounter in this thesis is the *Igusa cusp form* which is the unique weight 10 SMF on $Sp(2,\mathbb{Z})$. The Igusa cusp form is defined as

$$\Phi_{10}(\Omega) = qyp \prod_{\substack{(\ell \in \mathbb{Z}, m \geq 0, n > 0) \\ (\ell \in \mathbb{Z}_-, n = m = 0)}} \left(1 - q^n y^\ell p^m\right)^{2C(4mn - \ell^2)}, \tag{2.28}$$

where the coefficients C in the exponent, as will be expanded upon in Chapter 7, are related to the coefficients of the Fourier expansion of the elliptic genus of $K3$, which is in fact given in terms of $\phi_{0,1}(\tau, z)$ as defined in(2.20b). We shall expand upon this at a later stage. A final remark we wish to add on SMF's is regarding their constructions. We will not focus on the explicit construction of SMF's in this thesis. However, we do wish to remark on the construction of (2.28). There are two ways in which this can be constructed viz., the additive or *Saito–Kurokawa* lift [**Zag79**], or by the multiplicative or *Borcherds* lift [**GN96c, GN96a, GN96b**]. The Igusa cusp form as defined in (2.28) is in the product representation i.e., is constructed via the multiplicative lift. This is the representation that we shall employ for the remainder of this thesis.

2.4. Mock modular forms

We now turn to the concept of mock modular forms (MMF's). Mock modular forms are a relatively nascent field of mathematics first studied in detail in [**Zwe08**]. MMF's were first discovered by Srinivasa Ramanujan [**Ram88, Zag07**]. The nature of mock-modularity is that a holomorphic function transforms under the action of $SL(2,\mathbb{Z})$ almost, but not exactly, like a modular form. We shall broadly deal with two different kinds of mock modular forms: pure and mixed.

Definition 2.4.1 (Pure mock modular form and its shadow). *A (weakly holomorphic) pure mock modular form $h(\tau)$ is a form of weight $k \in \mathbb{Z}/2$ that is holomorphic on the UHP with at most exponential growth at cusps. To this form $h(\tau)$, we associate a function $g(\tau)$ which is the shadow of $h(\tau)$. This shadow is a holomorphic modular form $g(\tau)$ of weight $2-k$ that 'completes' the mock modular form as follows.*

The non-holomorphic Eichler integral of $g(\tau)$, denoted by $g^\star(\tau)$ is the solution to the differential equation

$$2^{2k-1}\pi^{k-1} i \tau_2^k \frac{\partial g^\star(\tau)}{\partial \bar{\tau}} = \overline{g(\tau)}, \tag{2.29}$$

such that the sum $\widehat{h}(\tau) := h(\tau) + g^\star(\tau)$ (the completion of $h(\tau)$) transforms like a holomorphic modular form of weight k. Since $h(\tau)$ is a holomorphic function on the UHP, it is also related to the shadow similar to (2.29) as

$$2^{2k-1}\pi^{k-1} i \tau_2^k \frac{\partial \widehat{h}(\tau)}{\partial \bar{\tau}} = \overline{g(\tau)}\,. \tag{2.30}$$

Since $g(\tau)$ is a holomorphic modular form, it admits a Fourier expansion of the form $g(\tau) = \sum_n b(n) q^n$. Note that the shadow should be optimally chosen. By this we mean that the nature of the form $g(\tau)$ depends on the nature of the mock modular form. The shadow has to be chosen in such a way that it satisfies the overall growth properties of the modular form following completion. In other words, the growth property of the shadow has to be optimally chosen so as for the completed form to have the correct growth [DMZ12]. The choice of the shadow can be chosen by setting

$$g^\star(\tau) = \begin{cases} \bar{b}_0 \dfrac{(4\pi\tau_2)^{-k+1}}{k-1} + \displaystyle\sum_{n>0} n^{k-1}\bar{b}_n q^{-n} \underbrace{\int_{4\pi n\tau_2}^{\infty} t^{-k}e^{-t}dt}_{\text{incomplete }\Gamma-\text{function}}\,, & k \neq 1 \\ -\bar{b}_0 \log(4\pi\tau_2) + \displaystyle\sum_{n>0} n^{k-1}\bar{b}_n q^{-n} \underbrace{\int_{4\pi n\tau_2}^{\infty} t^{-k}e^{-t}dt}_{\text{incomplete }\Gamma-\text{function}}\,, & k = 1 \end{cases}. \tag{2.31}$$

When the shadow $g(\tau)$ is a unary theta series of the form

$$f_u \tilde{=} \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} \epsilon(n) q^{\beta n^2} & \text{for some } \beta \in \mathbb{Q}_+,\ \epsilon \text{ is an even periodic function} \\ \displaystyle\sum_{n\in\mathbb{Z}} n\epsilon(n) q^{\beta n^2} & \text{for some } \beta \in \mathbb{Q}_+,\ \epsilon \text{ is an odd periodic function} \end{cases}, \tag{2.32}$$

the mock modular form is a mock theta function, as was first described in [Ram88]. We shall also define the notion of a mixed mock modular form as a mock modular form for which the completion of $h(\tau)$, $\hat{h}(\tau)$, is expressible as

$$\hat{h}(\tau) = h(\tau) + \sum_i f_i(\tau) g_i^\star(\tau), \tag{2.33}$$

where $f_i(\tau)$'s are modular forms of weight l and $g(\tau)$'s are modular forms of weight $2-k+l$. In other words, the shadow can be further broken down into product of modular forms. Mock modular forms make a wide appearance in physics. They appear as the partition function of the topologically twisted super Yang-Mills theory on $\mathbb{CP}^2$ [VW94, DPW20, Ale20]. One class of mock modular forms that will appear in this thesis is related to the Mathieu moonshine as in Chapter 5. We define

$$F_2^{(2)}(\tau) := \sum_{\substack{r>s>0 \\ r-s\in 2\mathbb{Z}+1}} (-1)^r s q^{\frac{rs}{2}} = q + q^2 - q^3 + q^4 - q^5 + \cdots\,. \tag{2.34}$$

The function

$$\begin{aligned} H(\tau) &:= 2\left(\frac{24F_2^{(2)}(\tau) - E_2(\tau)}{\eta(\tau)^3}\right) \\ &= 2q^{-1/8}\left(-1 + 45\ q + 231\ q^2 + 770\ q^3 + 2277\ q^4 + \cdots\right)\,, \end{aligned} \tag{2.35}$$

has Fourier expansion coefficients which are precisely the dimensions of the irreducible representations of the $\mathbf{M}_{24}$ group. The function (2.35) can also be related to the elliptic genus of $K3$ and this will become apparent and significant in Chapter 5.

2.5. Mock Jacobi forms

Mock Jacobi forms will be another of the main focal points of this thesis. Mock Jacobi forms are identical to Jacobi forms (2.2.1) in their ellipticity properties but their modular transformation properties are not exact on $SL(2,\mathbb{Z})$. They still exhibit a theta decomposition as in (2.15) but the $h_\ell(\tau)$ are mock modular forms of weight $k-\frac{1}{2}$. Mock Jacobi forms have a completion, similar to mock modular forms. The completion is manifest again in $h_\ell(\tau)$ as

$$\widehat{\phi}(\tau,z) = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} \widehat{h}_\ell(\tau)\vartheta_{m,\ell}(\tau,z)\,, \tag{2.36}$$

where the completion affects the vector–valued (mock) modular forms ($h_\ell(\tau)$ is completed to $\widehat{h}_\ell(\tau)$) in the theta decomposition of a mock Jacobi form. This can be appreciated by putting together equation (2.15) and Definition 2.4.1. Let $g_\ell(\tau)$ be the shadow of $h_\ell(\tau)$. The completion of the mock Jacobi form follows as

$$\widehat{\phi}(\tau,z) = \phi(\tau,z) + \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} g_\ell^\star(\tau)\vartheta_{m,\ell}(\tau,z)\,, \tag{2.37}$$

where $g_\ell^\star(\tau)$ is as defined in (2.31). Mock Jacobi forms appear in a wide range of problems as studied in this thesis. They first made their appearance (prior to the work done in [**Zwe08**]) in the study of $\mathcal{N}=2,4$ superconformal characters [**EOTY89, ET88b**]. These shall be introduced later in Chapter 5. More problem specific machinery concerning mock modular forms shall be introduced later as and when required. They appear in the counting function of $\frac{1}{4}$–BPS dyons in four dimensional theories with sixteen supercharges, as we shall see in Section 7.7. Although they do describe legitimate partition functions in physics, their thermodynamic interpretation in a general situation remains unclear [**DMZ12**]. Progress towards the origin of mock modularity in the context of the counting function for single center $\frac{1}{4}$–BPS black holes in $\mathcal{N}=4,\ d=4$ string theory has been made in [**MP18**].

2.6. Theta functions

We provide a very conservative introduction to theta functions in this section. We refer the reader to the classic texts [**Mum83, Mum84, MNN07**] for more literature on the theta functions. Consider the function $\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}$. This counts the number of ways in which a integer can be expressed as a sum of even squares. It is therefore the theta function associated to a lattice as the count of vectors of a particular norm $\langle x,x\rangle = x^2$. It can be represented as an infinite product as

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)\left(1+q^{n-\frac{1}{2}}\right)^2. \tag{2.38}$$

We can also introduce two more theta functions as

$$\theta_2(\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{n^2/2} = 2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)\,(1+q^n)^2\,. \tag{2.39a}$$

$$\theta_4(\tau) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2/2} = 2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)\left(1-q^{n-\frac{1}{2}}\right)^2. \tag{2.39b}$$

These are related to the Dedekind-eta function (2.7a) by

$$2\eta(\tau)^3 = \theta_4(\tau)\theta_3(\tau)\theta_2(\tau). \tag{2.40}$$

These three functions (2.38)(2.39a)(2.39b) are components of a vector–valued modular form on $SL(2,\mathbb{Z})$

$$\Theta(\tau) = \begin{pmatrix}\theta_2(\tau)\\ \theta_3(\tau)\\ \theta_4(\tau)\end{pmatrix}, \tag{2.41}$$

satisfying

$$\Theta(\tau) = \sqrt{\frac{i}{\tau}}\mathcal{S}\Theta\left(-\frac{1}{\tau}\right) = \mathcal{T}\Theta(\tau+1)\,, \tag{2.42}$$

with

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} e^{i\pi/4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.43}$$

The theta functions (2.38),(2.39a) and (2.39b) can be 'generalized' to the Jacobi theta functions defined below by demanding elliptic transformation properties. The Jacobi theta functions are expressible as an infinite sum or product as

$$\begin{aligned} \vartheta_1(\tau,z) &= -i\sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^{n-\frac{1}{2}} y^n q^{n^2/2} \\ &= -iq^{\frac{1}{8}}\left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)\,. \end{aligned} \tag{2.44a}$$

$$\begin{aligned} \vartheta_2(\tau,z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} y^n q^{n^2/2} \\ &= q^{\frac{1}{8}}\left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^n)\,. \end{aligned} \tag{2.44b}$$

$$\begin{aligned} \vartheta_3(\tau,z) &= \sum_{n\in\mathbb{Z}} y^n q^{n^2/2} \\ &= \prod_{n=1}^{\infty}(1-q^n)(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}})\,. \end{aligned} \tag{2.44c}$$

$$\begin{aligned} \vartheta_4(\tau,z) &= \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{n^2/2} \\ &= \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n-\frac{1}{2}})(1-y^{-1}q^{n-\frac{1}{2}})\,. \end{aligned} \tag{2.44d}$$

The theta functions (2.39a), (2.38), (2.39b) are related to the Jacobi theta functions by

$$\theta_i(\tau) = \vartheta_i(\tau, z=0), \quad i = 2,\ 3,\ 4\,. \tag{2.45}$$

2.7. Rademacher expansion and Rademacher series

We have thus far only briefly mentioned the importance of understanding the coefficients in the Fourier expansion of an automorphic form in their classification in Section 2.1.1 and Section 2.2.1. Computing said coefficients is of physical significance and as we shall see, in the context of the partition functions of conformal field theories, the Fourier coefficients capture the degeneracies of the system. However, precise computation of Fourier expansion coefficients is not a trivial task. While there are many techniques to compute the coefficients of the Fourier expansion, we shall focus on the Hardy–Ramanujan–Rademacher method (Rademacher expansion)

[**Rad43, Rad38, Rad12, RZ38, HR18a, HR18b**] here. We refer the reader to [**Rad12, DF11, CD14**] for more details on the Rademacher expansion that are pertinent but not pedagogically required to understand this thesis.
The Rademacher expansion provides a powerful tool to reconstruct the Fourier coefficients of modular forms. For the case of Jacobi forms, one requires the use of generalized Rademacher series [**DMMV00, FR17**]. We shall consider it here for forms of weights $w + \frac{1}{2} \leq 0$ on the modular group $SL(2,\mathbb{Z})$ and a generic multiplier system $\psi(\gamma)$. The Rademacher expansion only requires knowledge of the modular properties of $h_\ell(\tau)$ and values of a finite number of coefficients of negative powers of q in the q–expansion in order to compute the Fourier coefficients $c(n,\ell)$ with $\Delta \geq 0$. We define the terminology of *polar coefficients/terms* i.e., the terms with negative powers of q in the Fourier expansion

$$h_\ell(\tau) \;=\; \sum_{\widetilde{\Delta}<0} c(\tilde{n},\tilde{\ell})\, q^{\widetilde{\Delta}/4m} + \sum_{\Delta\geq 0} c(n,\ell)\, q^{\Delta/4m}\,. \tag{2.46}$$

The Rademacher expansion for the Fourier coefficients of $h_\ell(\tau)$ is given by
(2.47)

$$c(n,\ell) \;=\; 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\tilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}\\ \widetilde{\Delta}<0}} c(\tilde{n},\tilde{\ell})\, \frac{\mathrm{KLS}\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m}\,;k,\psi\big)_{\ell\tilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{\frac{1-w}{2}} I_{1-w}\Big(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\Big)\,,$$

where $I_\rho(x)$ is the I-Bessel function of weight ρ. It has the following integral representation for $x \in \mathbb{R}^*$,

$$I_\rho(x) \;=\; \frac{1}{2\pi\mathrm{i}} \left(\frac{x}{2}\right)^{\rho} \int_{\epsilon-\mathrm{i}\infty}^{\epsilon+\mathrm{i}\infty} t^{-\rho-1}\, e^{t+\frac{x^2}{4t}}\, \mathrm{d}t\,, \tag{2.48}$$

and has the asymptotic behaviour

$$I_\rho(x) \underset{x\to\infty}{\sim} \frac{e^x}{\sqrt{2\pi x}}\Big(1-\frac{\mu-1}{8x}+\frac{(\mu-1)(\mu-3^2)}{2!(8x)^3}-\frac{(\mu-1)(\mu-3^2)(\mu-5^2)}{3!(8x)^5}+\ldots\Big)\,, \tag{2.49}$$

with $\mu = 4\rho^2$. In (2.47), $\mathrm{KLS}\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m}\,;k,\psi\big)_{\ell\tilde{\ell}}$ is the generalized Kloosterman sum [**DMMV00**]

$$\mathrm{KLS}(\mu,\nu\,;k,\psi)_{\ell\tilde{\ell}} \coloneqq \sum_{\substack{0\leq h<k\\ (h,k)=1}} e^{2\pi\mathrm{i}\left(-\frac{h}{k}\mu+\frac{h'}{k}\nu\right)}\, \psi(\gamma)_{\ell\tilde{\ell}}\,, \tag{2.50}$$

with $\gamma = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix} \in SL(2,\mathbb{Z})$ and $hh' \equiv -1 \pmod{k}$. While all this deals with modular and Jacobi forms on $SL(2,\mathbb{Z})$, we shall revisit the case of Rademacher expansions for mock modular/Jacobi forms in Chapter 7 and Chapter 8.

2.8. Automorphic forms in this thesis, and in physics

Automorphic forms, owing to their high symmetry, appear in many different aspects of physics. From the point of view of this thesis, automorphic forms are related to the count of BPS states and special cycles on Calabi–Yau manifolds (Gromov–Witten/Donaldson–Thomas invariants). The first part of this thesis is devoted to moonshine phenomena which is the study of the symmetries of some of these BPS automorphic forms and their connection to finite simple sporadic groups. The second

part of this thesis focuses on precision counting and matching of the coefficients in the Fourier expansion of automorphic forms on $SL(2, \mathbb{Z})$ and $Sp(2, \mathbb{Z})$, where the automorphic forms are partition functions in string compactification that count the number of BPS states/black hole degeneracies. To a large extent, the most prominent application of automorphic forms to physics is in the study of black hole partition functions.
Another string theoretic application of automorphic forms is in the study of constraints on string scattering amplitudes where higher genus contributions to the string amplitude can be calculated from studying automorphic forms on a genus–g surface related to the moduli space of those genus–g surfaces [**GRV10**, **DGV15**, **Log19**].[9] Automorphic forms and function appear in a wide range of other fields in physics. The study of automorphic $L-$functions appear in the study of dynamical systems and chaos [**Sar93**, **Sar97a**, **Sar97b**]. Automorphic forms are central in the Langlands program which is also considered to have far reaching consequences in conformal field theory and quantum field theory [**Fre07**, **KW07**].

[9]We thank Kimberly Logan for discussions during the workshop on *Automorphic structures in string theory* for bringing attention to a more substantial body of mathematical literature relating automorphic forms to string amplitudes.

CHAPTER 3

Calabi–Yau manifolds

Overview of this chapter

In this chapter, we provide an overview of the required concepts in the study of Calabi–Yau manifolds. Calabi–Yau (CY) manifolds are an important tool in the study of compactification of string theory to four spacetime dimensions [**CHSW85**] and their geometric properties allow for us to control the structure of the resulting theory [**Gre96**]. For more detailed studies of complex manifolds, we refer the reader to [**Huy05**]. For Kähler geometry, we refer the reader to [**Mor07, Huy16**] and for Calabi–Yau geometry, we refer the reader to [**Hub92, GHJ12**]. Standard references on more advanced topics include [**HKK$^+$03, CK00, ABC$^+$09**]. We shall comment on the physical importance of CY manifolds later.

3.1. Complex geometry

Consider a real manifold manifolds $\mathcal{M}_{2d}$ of dimension $2d$. A real manifold of even dimension allows us to define a complex manifold as follows.

Definition 3.1.1 (Complex manifolds). *Let $\{U_i\}$ be an open covering of the manifold $\mathcal{M}_{2d}$. Let there be a homeomorphism $\phi_i : U_i \to \mathbb{C}^d$ on to an open set on $\mathbb{C}^d$. A manifold $(\mathcal{M}, \{U_i, \phi_i\})$ is complex (of complex dimension $\dim_{\mathbb{C}} = \frac{\dim_{\mathbb{R}}}{2}$) if $\forall\ U_i \cap U_j \neq \emptyset$, $\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is a holomorphic transition function. The transition functions, being holomorphic, satisfy the Cauchy–Riemann equation*

$$\bar{\partial}_{\bar{k}} \phi_{ij} = \frac{\partial}{\partial \bar{z}^k} \phi_{ij} = 0\ \forall\ k\ , \tag{3.1}$$

where $z^k = x^k + iy^k$ are the complex coordinates.

Henceforth, the derivative ∂_k will refer to a derivative with respect to z^k, while the derivative $\bar{\partial}_{\bar{k}}$ will refer to a derivative with respect to $\bar{z}^k$.

Examples:

(a) The manifold $\mathbb{C}^d$ is a $d-$ dimensional complex manifold.
(b) The 2–torus as seen in Chapter 2.
(c) Another complex manifold that we shall encounter is the complex projective space $\mathbb{CP}^n$ which is constructed as follows. Consider the set of points up to a complex identification $(z_0, z^1, \cdots, z^n) \sim \lambda(z_0, z^1, \cdots, z^n)$, $\lambda \in \mathbb{C}$. Such an identification acts on a degree p polynomial which can be expressed as using homogeneous coordinates

$$f(z_0, z^1, \cdots, z^n) = z^p g(z_0/z^n, z^1/z^n, \cdots, z^{n-1}/z^n)\,. \tag{3.2}$$

Since the transition functions between f and g are holomorphic, a complex projective space $\mathbb{CP}^n$ is a complex manifold of $\dim_{\mathbb{C}} = n$.

Every complex manifold is a real manifold, while the converse is not true. An even dimensional real manifold can be complexified iff it admits a complex structure.

DEFINITION 3.1.2 (Almost complex structure). *Consider a real manifold $\mathcal{M}_{2d}$. An almost complex structure J on $\mathcal{M}$ is a smooth tensor field J such that*

$$-J_a{}^b J_b{}^c = \delta_a{}^c \,. \tag{3.3}$$

A $\mathcal{M}_{2d}$ with an almost complex structure is known as an almost complex manifold.

DEFINITION 3.1.3 (Nijenhuis tensor). *The Nijenhuis tensor $N_J(x,y)$ of two vector fields x, y is defined as*

$$N_J(x,y) = [Jx, Jy] - J[x, Jy] - J[Jx, y] - [x,y]\,, \tag{3.4}$$

where $[\cdot,\cdot]$ is the standard Lie bracket between two vector fields.

DEFINITION 3.1.4 (Complex manifold). *An almost complex manifold with an almost complex structure J such that the Nijenhuis tensor (3.4) identically vanishes is known as a complex manifold.*

REMARK 3.1.1. *The definitions Definition 3.1.1 and Definition 3.1.4 are equivalent. This is the statement of the Newlander–Nirenberg theorem* [**Voi**].

3.2. (Co-)Homology

Consider $k-$forms on a real manifold $\mathcal{M}$. These forms are given in terms of the smooth sections of the the exterior power of the cotangent bundle $\Lambda^k T^*\mathcal{M}$ and are therefore totally anti-symmetric tensors of the type $(0,k)$. The dimension of $\Lambda^k T^*\mathcal{M}$ is $\begin{pmatrix} m \\ k \end{pmatrix}$ where m is the dimension of the manifold. The space of k–forms is denoted as $\Omega_k(\mathcal{M})$. Consider a $k-$ and an $l-$form on $\mathcal{M}$ as

$$\begin{aligned} \alpha_k &= \alpha_{i_1\cdots i_k}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}\,, \\ \beta_l &= \beta_{i_1\cdots i_l}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_l}\ . \end{aligned} \tag{3.5}$$

Then the exterior product (or wedge product) between a k–form and l–form is a $k+l$–form

$$\alpha_k \wedge \beta_l = \alpha_{i_1\cdots i_k}\beta_{i_{k+1}}\cdots{}_{i_{k+l}}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \wedge dx^{i_{k+1}} \cdots \wedge dx^{i_{k+l}}\,. \tag{3.6}$$

The exterior derivative $d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ and is given by

$$d\alpha = \frac{\partial}{\partial x^{i_0}} \alpha_{i_0, i_i, \cdots i_{k+1}} dx^{i_0} \wedge x^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}\ , \tag{3.7}$$

such that $d^2\alpha = 0$ for any differential k–form. A k–form is *closed* if its exterior derivative vanishes i.e., $d\alpha = 0$. A k–form is *exact* if it is the exterior derivative of a $k-1$–form i.e., $\alpha = d\beta$, $\beta \in \Omega^{k-1}(\mathcal{M})$. Since $d^2\beta = 0$, then $d\alpha = 0$ which means that every exact form is closed.

Let $A_0,\ A_1, \cdots$ be Abelian groups connected by homomorphisms $d_n : A_n \to A_{n+1}$ such that $d_n \circ d_{n-1} = 0, \forall n$. A cochain complex is then the sequence

$$0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} \cdots \ . \tag{3.8}$$

This allows us to define cohomology groups H^n

$$H^n = \frac{\mathrm{Ker}(d_n : A_n \to A_{n+1})}{\mathrm{Image}(d_n : A_{n-1} \to A_n)} \ . \tag{3.9}$$

An important example of a cohomology on a manifold is the *de Rham cohomology*. To define this, we use the action of the exterior derivative to define the de Rham complex as

$$0 \xrightarrow{d} \Omega^0(\mathcal{M}) \xrightarrow{d} \Omega^1(\mathcal{M}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(\mathcal{M}) \xrightarrow{d} 0 \ , \tag{3.10}$$

and the de Rham cohomology groups $H^k_{dR}(\mathcal{M}, \mathbb{R})$

$$H^k_{dR}(\mathcal{M}, \mathbb{R}) = \frac{\mathrm{Ker}(d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M}))}{\mathrm{Image}(d : \Omega^{k-1}(\mathcal{M}) \to \Omega^k(\mathcal{M}))} \ . \tag{3.11}$$

Elements of the cohomology may be related to create an equivalency classes within the cohomology and this is known as the cohomology class. Two elements (α, β) of the cohomology H^k_{dR} are in the same cohomology class, $[\alpha]$, if there exists $\lambda \in \Omega^{k-1}(\mathcal{M})$ such that $\alpha = \beta + d\lambda$.

Definition 3.2.1 (Betti number). *The dimension of the $k-$th de Rham cohomology group $H^k_{dR}(\mathcal{M}, \mathbb{R})$ is the $k-$th Betti number, b^k, i.e.,*

$$b^k = \dim \left(H^k_{dR}(\mathcal{M}, \mathbb{R}) \right) . \tag{3.12}$$

The Betti numbers and cohomological data can be used to compute the Euler characteristic of the manifold.

Definition 3.2.2 (Euler characteristic). *The Euler characteristic $\chi(\mathcal{M})$ of a manifold is a topological invariant*

$$\chi(\mathcal{M}) = \sum_{i=0}^{n} (-1)^i b^i \, . \tag{3.13}$$

For compact manifolds, without boundaries, (3.13) is given by $\chi(\mathcal{M}) = 2 - 2g$, where g is the number of "holes" or the genus of the manifold.

For the case of a complex manifold of dimension $\mathcal{M}$ of dimension n, there is a further subtlety that arises from the holomorphic and anti–holomorphic separation of the cotangent space. In other words, we need to consider the holomorphic and anti–holomorphic sections which give rise to p, q–forms. The space of k–forms can be decomposed into a direct sum of spaces of $\Omega^{(p,q)}(\mathcal{M})$ forms as

$$\Omega^k(\mathcal{M}) = \bigoplus_{i=0}^{k} \Omega^{i,k-i}(\mathcal{M}) \, , \tag{3.14}$$

so the interpretation that we have here is that a p, q–form is a differential form that a holomorphic p–form and an anti-holomorphic q–form. An exterior derivative on a complex manifold of dimension n can also be decomposed into holomorphic and anti-holomorphic parts as

$$
\begin{aligned}
d &= \partial + \bar{\partial}\,, \\
\partial &: \Omega^{p,q}(\mathcal{M}) \to \Omega^{p+1,q}(\mathcal{M})\,, \\
\bar{\partial} &: \Omega^{p,q}(\mathcal{M}) \to \Omega^{p,q+1}(\mathcal{M})\,.
\end{aligned}
\tag{3.15}
$$

The requirement that $d^2 = 0$ implies that $\partial^2 = 0,\ \bar{\partial}^2 = 0,\ \partial\bar{\partial} + \bar{\partial}\partial = 0$. We can now begin defining the Dolbeault cohomology by first defining the Dolbeault complex as

$$
0 \xrightarrow{\bar{\partial}} \Omega^{p,0}(\mathcal{M}) \xrightarrow{\bar{\partial}} \Omega^{p,1}(\mathcal{M}) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{p,n}(\mathcal{M}) \xrightarrow{\bar{\partial}} 0\,,
\tag{3.16}
$$

or equivalently using the holomorphic exterior derivative as

$$
0 \xrightarrow{\partial} \Omega^{0,q}(\mathcal{M}) \xrightarrow{\partial} \Omega^{1,q}(\mathcal{M}) \xrightarrow{\partial} \cdots \xrightarrow{\bar{\partial}} \Omega^{n,q}(\mathcal{M}) \xrightarrow{\partial} 0\,.
\tag{3.17}
$$

From this, the Dolbeault cohomology group is defined as

$$
H^{p,q}_{\bar{\partial}} = \frac{\mathrm{Ker}(\bar{\partial} : \Omega^{p,q}(\mathcal{M}) \to \Omega^{p,q+1}(\mathcal{M}))}{\mathrm{Image}(\bar{\partial} : \Omega^{p,q-1}(\mathcal{M}) \to \Omega^{p,q}(\mathcal{M}))}\,,
\tag{3.18}
$$

while for the holomorphic exterior derivative, we have

$$
H^{p,q}_{\partial} = \frac{\mathrm{Ker}(\partial : \Omega^{p,q}(\mathcal{M}) \to \Omega^{p+1,q}(\mathcal{M}))}{\mathrm{Image}(\partial : \Omega^{p-1,q}(\mathcal{M}) \to \Omega^{p,q}(\mathcal{M}))}\,.
\tag{3.19}
$$

The Dolbeault cohomology is not universally defined on a complex manifold and depends on the choice of the complex structure. Similar to the Betti numbers (3.12), we can define the Hodge numbers of a complex manifold.

Definition 3.2.3 (Hodge numbers). *The Hodge number $h^{p,q}$ is the complex dimension of the $H^{p,q}_{\bar{\partial}}$ cohomology group i.e.,*

$$
h^{p,q}(\mathcal{M}) = \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(\mathcal{M})\,.
\tag{3.20}
$$

The Hodge numbers of a complex manifold can be used to define the (holomorphic) Euler characteristic as

$$
\chi^p(\mathcal{M}) = \sum_{q\geq 0} (-1)^q h^{p,q}(\mathcal{M})\ ,
\tag{3.21}
$$

where $\mathcal{M}$ is a complex manifold. The Hodge numbers can be arranged in a rather nice representation, known as the Hodge diamond, as follows

$$
D_{\mathcal{M}_d} = \begin{array}{ccccccc}
 & & & h^{d,d} & & & \\
 & & h^{d,d-1} & & h^{d-1,d} & & \\
 & \iddots & & \vdots & & \ddots & \\
h^{d,0} & & \cdots & & \cdots & & h^{0,d} \\
 & \ddots & & \vdots & & \iddots & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}\,,
\tag{3.22}
$$

where in the above expression d is the complex dimension of the manifold. We now define the additional metric structure on the manifold. This leads us to the concept of Hermitian and Kähler manifolds.

3.3. Kähler manifolds

DEFINITION 3.3.1 (Hermitian metric). *Let $\mathcal{M}_d$ be a complex manifold with complex dimension d with a complex structure J. A hermitian metric g is an inner product defined at every point $p \in \mathcal{M}_d$ such that $g : T^{1,0}\mathcal{M}_d \otimes T^{0,1}\mathcal{M}_d \to \mathbb{C}$ such that it is positive definite. The structure of the Hermitian metric implies that g is a Riemannian metric on $\mathcal{M}_d$ the metric can be expressed as a $1,1$–form*

$$\omega^{1,1}(v, w) = g(v, Jw), \quad v \in \Gamma(T^{1,0}\mathcal{M}_d), \quad w \in \Gamma(T^{0,1}\mathcal{M}_d)\,. \tag{3.23}$$

DEFINITION 3.3.2 (Kähler metric and manifold). *Consider a complex manifold $\mathcal{M}_d$ with complex structure J and a Hermitian metric g with associated Hermitian $1,1$–form ω. The metric g is Kähler if the Hermitian $1,1$–form is closed i.e., $d\omega = 0$. In this case, the $1,1$–form ω is called the Kähler form and the manifold $(\mathcal{M}_d, J, g)$ is a Kähler manifold.*

REMARK 3.3.1. *The Kähler metric may also be written as*

$$\omega = i \; g_{\mu\bar{\nu}} \; dz^{\mu} \wedge d\bar{z}^{\bar{\nu}} \; . \tag{3.24}$$

The Kähler condition of requiring $d\omega = 0$ translates into

$$\partial_\mu g_{\nu\bar{\kappa}} = \partial_\nu g_{\mu\bar{\kappa}}, \quad \bar{\partial}_{\bar{\nu}} g_{\mu\bar{\kappa}} = \bar{\partial}_{\bar{\kappa}} g_{\mu\bar{\nu}} \; . \tag{3.25}$$

Since the Kähler form is closed, it determines an element in the second de Rham cohomology known as the *Kähler class*. The Kähler $1,1$–form ω admits a local function K such that

$$\begin{aligned} \omega &= \frac{i}{2}\partial_\mu \bar{\partial}_{\bar{\nu}} K \,, \\ g_{\mu\bar{\nu}} &= \partial_\mu \bar{\partial}_{\bar{\nu}} K \,. \end{aligned} \tag{3.26}$$

Here, K is the Kähler potential. For a $d-$dimensional Kähler manifold $\mathcal{M}_d$, the volume form may be defined in terms of the Kähler form as

$$\mathrm{vol}(\mathcal{M}_d) = \frac{1}{d!}\int_{\mathcal{M}} \omega^d. \tag{3.27}$$

One can view this from the view point of the Dolbeault cohomology as well (3.18). Since there are equivalence classes within a Dolbeault cohomology group, it is therefore not an obvious statement to say that all elements of a Dolbeault cohomology $H^{1,1}_{\bar{\partial}}$ give rise to Kähler forms. The necessary condition for an equivalence class in the Dolbeault cohomology to give rise to a Kähler form is to demand positivity of the metric. This results in a 'conical' structure endowed on those classes which lift up to Kähler forms. This is known as the *Kähler cone*.

DEFINITION 3.3.3 (Hodge star). *Let α, β be complex k–forms on a Kähler manifold $\mathcal{M}_d$. Let there be an inner product defined by*

$$(\alpha, \beta) = \alpha_{\mu_1 \cdots \mu_k} \bar{\beta}_{\bar{\nu}_1 \cdots \bar{\nu}_k} g^{\mu_1 \bar{\nu}_1} \cdots g^{\mu_k \bar{\nu}_k} . \tag{3.28}$$

The Hodge star $\star_H$ is an isomorphism $\star_H : \Lambda^k T^* \mathcal{M} \to \Lambda^{d-k} T^* \mathcal{M}$ such that given a complex k–form, β, $\star_H \beta$ is the unique $d-k$–form such that

$$\alpha \wedge \star_H \beta = (\alpha, \beta)\ d\mu, \tag{3.29}$$

where $d\mu$ is the volume form in the corresponding basis for the metric g.

The action of a Hodge star operator is therefore the map $\star_H : \Omega^{p,q}(\mathcal{M}) \to \Omega^{d-p,d-q}(\mathcal{M})$.

DEFINITION 3.3.4 (Ricci form). *The Ricci tensor (a Ricci $1,1$–form) is a form $\mathcal{R}$ such that*

$$\mathcal{R} = i R_{\mu\bar{\nu}}\ dz^{\mu} \wedge d\bar{z}^{\bar{\nu}} = \frac{i}{2} d(\partial - \bar{\partial}) \log \det g, \tag{3.30}$$

where d is the exterior derivative operator in this case.

3.4. Homology and cycles

Let us define an operator ∂ that maps a compact submanifold $B \subset M$ to its boundary ∂B. If $\partial B = 0$, we say that B has no boundary. If $\beta = \partial \alpha$, then we say that β is the boundary of some submanifold $\alpha \subset M$. In order to define a homology, we define a chain complex $C(M)$. The chain complex is a sequence of Abelian groups C_i that is related by homomorphisms of the boundary operator as

$$\partial_n : C_n \to C_{n-1}. \tag{3.31}$$

This produces a sequence

$$\cdots \xrightarrow{\partial_{n+1}} C_n(M) \xrightarrow{\partial_n} C_{n-1}(M) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} C_0(M) \xrightarrow{\partial_0} 0. \tag{3.32}$$

From the above sequence, it is clear that the composition of two consecutive boundary operators is trivial i.e., $\partial_n \circ \partial_{n+1} = 0$. In other words, the boundary of a boundary of a submanifold is trivial.

DEFINITION 3.4.1 (Homology group). *The $n-$th homology group $H_n(M)$ is given by*

$$H_n(M) = \frac{Ker\,(\partial_n : C_n(M) \to C_{n-1}(M))}{Image\,(\partial_{n+1} : C_{n+1} \to C_n)}. \tag{3.33}$$

Elements of Image $(\partial_{n+1} : C_{n+1} \to C_n)$ are called boundaries, while elements of Ker $(\partial_n : C_n(M) \to C_{n-1}(M))$ are known as cycles. Elements of a homology group are called homology classes and each homology class is an equivalence class over cycles. Therefore, two cycles in the same homology class are homologous. A homology group $H_n(M)$ counts essentially how many $n-$dimensional holes there are in a manifold M.

THEOREM 3.4.1 (de Rham). *$H^k(M)$, the $k-$th cohomology group of M and $H_k(M)$, the $k-$th homology group of M are isomorphic to each other.*

REMARK 3.4.1. *Consider a d–dimensional manifold, M. Consider a k–form $\alpha \in H^k(M)$ and an $d-k$–form $\beta \in H^{d-k}(M)$. Using the wedge product, we get an d–form $\alpha \wedge \beta$ as*

$$H^k(M) \times H^{n-k}(M) \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta. \tag{3.34}$$

Since the inner product of differential forms as above is non-degenerate, $H^k(M) \cong H^{d-k}(M)$. This is known as Poincaré duality.

Before we introduce the notion of Calabi–Yau manifolds, we define two quantities.

DEFINITION 3.4.2 (Chern Class). *Consider a complex vector bundle $E \xrightarrow{\pi} M$. Let A be the connection on E with curvature 2–form $F = dA + A \wedge A$. The total Chern class of E, $c(E)$, is*

$$c(E) = \det\left(1 + \frac{i}{2\pi}F\right). \tag{3.35}$$

The Chern classes $c_i(E)$ are the expansion of the Chern class $c(E)$ of even cohomology $c_i(E) \in H^{2i}(M, \mathbb{R})$ as

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots c_{d/2}(E), \tag{3.36}$$

where d is the real dimension of the manifold M.

The Chern class c_i of a d–dimensional manifold vanishes for $i > \dfrac{m}{2}$. The zeroth Chern class of a bundle is always 1. The first Chern class is given by

$$c_1(E) = \frac{i}{2\pi}\mathrm{Tr}F. \tag{3.37}$$

DEFINITION 3.4.3 (Holonomy group). *Let $\mathcal{M}_d$ be a d–dimensional Riemannian manifold with the metric g and a connection ∇. Consider the set of all loops around a point $p \in \mathcal{M}_d$. Any vector in the tangent space at p, $T_p\mathcal{M}_d$, can be parallel transported along the set of loops using the connection ∇. This defines a set of linear and invertible transformations from the tangent space at p onto itself. This set of transformations is endowed with a group structure and this is known as the 'Holonomy' group of the manifold.*

3.5. Calabi–Yau manifolds

Calabi–Yau manifolds were first conjectured in [**Cal54**, **Cal57**] and proven to exist in [**Yau78**] and are one of the key elements in the modern string theorist's toolkit.

Definition 3.5.1 (Calabi–Yau manifold). *A Calabi–Yau (CY) manifold X (also denoted CY_d) is a $2d$–dimensional real manifold that is compact and Kähler whose first Chern class $c_1(X) = 0$. Instead of demanding the vanishing of the first Chern class, we may also define a CY manifold equivalently as a compact, Kähler manifold satisfying*

(1) *Ricci flatness i.e., $\mathcal{R}(X) = 0$, where $\mathcal{R}$ is as defined in (3.30).*
(2) *The holonomy group of X is a subgroup of $SU(d)$.*
(3) *The canonical bundle is trivial.*
(4) *$\exists$ a nowhere vanishing d–form, $\Omega^{d,0}$.*

Remark 3.5.1. *In the above definition, we have stated that a CY manifold is compact. However, the Calabi conjecture has been generalized to non–compact cases and the existence of non-compact Calabi–Yau's was proven in* [**TY90, TY91**].

The simplest example of a CY manifold is the T^2 which is the unique CY_1. Its Hodge diamond (3.22) is

$$D_{T^2} = \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} . \tag{3.38}$$

From the context of string theory, CY_3 are of main significance. These manifolds have a Hodge diamond given by

$$D_{CY_3} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} , \tag{3.39}$$

where the symmetric structure of the Hodge diamond is due to

(1) Complex conjugation: $h^{p,q} = h^{q,p}$.
(2) Hodge duality: $h^{p,q} = h^{d-q,d-p}$.
(3) Mirror symmetry: $h^{p,q} = h^{3-p,q}$ [**HKK+03**]. This is a generalization of Serre duality (in a homological sense) which is $h^{p,0} = h^{d-p,0}$.

The computation of the full Hodge diamond for a CY_3 can be found in any standard string theory reference such as [**BLT13**] and we refer the reader to this book for the basic details of this calculation.

Definition 3.5.2 (Hyper–Kähler manifold). *A manifold $\mathcal{M}_{4k}$, $k \in \mathbb{Z}$ of real dimension $4k$ that is Kähler and has a holonomy group that is $Sp(2k) \subset SU(2k)$ is a hyper–Kähler manifold. A hyper–Kähler manifold is also Ricci flat and is necessarily a Calabi–Yau manifold.*

The most relevant example of a hyper–Kähler manifold that we shall discuss in this thesis is the $K3$ surface, which shall be discussed in Section 3.7. In addition to the $K3$ surface, we shall focus on a variety of CY manifolds from the viewpoint of the count of BPS states (or special cycles). Therefore, much of the additional literature in building up CY_d for $d > 2$ will not be reviewed here. We point the reader towards other sources such as [**Hub92**, **Haa02**, **HKK$^+$03**, **CK00**, **Gre96**, **GHJ12**] for more information regarding Calabi–Yau manifolds.

3.6. Moduli space of Calabi–Yau manifolds

The moduli space of CY manifolds is the concept that there exists a space of parameters that parameterize a given CY manifold, and changing these parameters smoothly gives rise to different Calabi–Yau geometries.[1] To see what this moduli space for a CY manifold is, recall that $\mathcal{R} = 0$ for a CY manifold which has an endowed metric g. We can ask as to what the possible deformations of g are, such that the Ricci form still vanishes [**CdlO91**]. The allowed deformations are given by the equation

$$\delta g = \delta g_{ij} dz^i \wedge dz^j + \delta g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} + c.c.\,. \tag{3.40}$$

Requiring that (3.40) make the Ricci form vanish implies that $d\bar{z}^{\bar{j}}$ is a harmonic form and is an element of the $H^{1,1}_{\bar{\partial}}(\mathcal{M})$ Dolbeault cohomology. In fact, the Ricci flat deformations are controlled by cohomology classes of the manifold. The two cohomology groups are the $H^{1,1}_{\bar{\partial}}$ and $H^{2,1}_{\bar{\partial}}$ Dolbeault cohomology groups. The deformations with pure indices in (3.40) do not preserve the Hermitian structure. The Hermitian structure can be restored using coordinate transformations that are not holomorphic. This means that these deformations can be made Hermitian with a different choice of complex structure i.e., a new set of coordinates that are not holomorphic functions of the original coordinate system. These are the deformations associated to the elements of $H^{2,1}_{\bar{\partial}}$ and are known as complex structure deformations. The deformations with mixed indices are deformations of the Kähler class in $H^{1,1}_{\bar{\partial}}$. The moduli space of Calabi–Yau manifolds is therefore controlled by the fact that there are smooth deformations of the complex structure and Kähler class of the CY manifold.

3.7. K3 surfaces

The most relevant CY manifold from the context of this thesis is the $K3$ surface. We refer the reader to [**Huy16**] for more details on $K3$ surfaces. From the point of view of $\mathcal{N} = (4,4)$ superconformal field theory, we refer the reader to [**NW01**] while from a string theory point of view, standard references are [**Asp96**, **AM96**].
$K3$ surfaces are the non-trivial examples of CY_2, with the trivial example being T^4.

Definition 3.7.1 ($K3$ surface). *A $K3$ surface is a CY_2 (i.e., a 4 real dimensional manifold) such that $h^{1,0}(X) = 0$.*

The Hodge diamond for $K3$ is given by

$$D_{K3} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}\,. \tag{3.41}$$

[1] From the point of view of algebraic geometry, these moduli spaces are usually varieties and often Calabi–Yau manifolds are referred to as Calabi–Yau varieties.

From the above Hodge diamond, we have that $b_2(K3) = 22$ which means that the homology group $G \cong \mathbb{Z}^{22}$ and therefore the lattice associated to $K3$ is a 22–dimensional lattice. The lattice of $K3$ has some signature $\Lambda^{m,n}$ with $m+n=22$. From the index of $K3$, we have that $\mathrm{ind}(K3) = n - m = -16$, where

$$\mathrm{ind}(K3) = -\int_{K3} \frac{2}{3}\, c_2(K3), \quad c_2(K3) = 24\,. \tag{3.42}$$

This gives us a lattice of signature $H_2(K3,\mathbb{Z}) \cong \Lambda^{19,3}_{K3}$. Therefore the choice of complex structure is the choice of constructing $\Lambda^{19,3} \subset \mathbb{R}^{19,3}$. The moduli space of Ricci flat metrics is a symmetric space given by

$$\mathcal{M}_{K3} = O\left(\Lambda^{19,3}\right) \backslash O(19,3;\mathbb{R}) / \left(O(19,\mathbb{R}) \times O(3,\mathbb{R})\right). \tag{3.43}$$

As a remark, the moduli space of algebraic $K3$'s (i.e., a $K3$ surface which is an embedding into a projective space) in full generality depends on the number of lines (pairs of fixed points) and the moduli space varies accordingly depending on how the $K3$ space is parameterized. The generalized moduli space depends on the *Picard rank* or Picard number of the $K3$ surface. The Picard rank is the rank of an Abelian group of algebraically equivalent divisors, known as the *Néron–Severi group*. A final comment that we wish to add is to make a connection with the other CY_2, viz., the torus T^4. There are points in the moduli space of $K3$ surfaces where $K3 \simeq T^4/\mathbb{Z}_2$.[2]

3.8. Invariants, BPS states and Calabi–Yau manifolds

The main reason why we are interested in CY manifolds in this thesis have to do with their geometric (pseudo-)invariants and their physical interpretations. An invariant of a manifold $\mathcal{M}$ is a function or scalar quantity that does not change with deformations of the manifold i.e., when one moves to a different point in the moduli space of the manifold. There are exceptions to this and we shall discuss them at a later stage when we study wall crossing phenomena.

3.8.1. The elliptic genus of a Calabi–Yau manifold. Let us first go over the concept of an elliptic genus from the point of view of a complex manifold. The Euler characteristic of a manifold $\chi(M)$ is one of the most elementary indices that we consider and is defined as the alternating sum of the dimensions of the cohomology groups of a manifold. For a manifold M, the Euler characteristic

$$\chi(M) = \sum_i (-1)^i \dim H^i(M)\ . \tag{3.44}$$

From the Hirzebruch–Riemann–Roch theorem [Hir88], it is known that the Chern class of M can be decomposed as

$$c(M) = \prod_i^{\dim M} (1 + x_i)\ , \tag{3.45}$$

[2] Other orbifolds of T^4 are also possible.

where x_i's are the eigenvalues of $\frac{i}{2\pi}F$. In this notation, the Euler characteristic is simply

$$\chi(M) = \int_M \prod_i^{\dim M} x_i \,. \tag{3.46}$$

For complex manifolds, there is a complex phase parameter multiplying x_i in (3.46) as

$$\chi(M,y) = \int_M \prod_i^{\dim M} x_i \frac{1-ye^{-x_i}}{1-e^{-x_i}}, \quad y = e^{2\pi i z}\,, \tag{3.47}$$

and $\lim_{z\to 0}\chi(M,y) = \chi(M)$. The generalization of (3.47) to include modular invariance is precisely the elliptic genus. The elliptic genus is defined as

$$\mathbf{EG}_M(\tau,z) = \int_M \prod_i^{\dim M} \left(x_i \frac{1-ye^{-x_i}}{1-e^{-x_i}} \prod_{j=1}^{\infty} \frac{1-q^j y e^{-x_i}}{1-q^j e^{-x_i}} \frac{1-q^j y^{-1} e^{-x_i}}{1-q^j e^{-x_i}} \right), \quad q = e^{2\pi i \tau}. \tag{3.48}$$

Naturally, we still have $\lim_{z\to 0}\mathbf{EG}_M(\tau,z) = \chi(M)$. The elliptic genera for CY manifolds are usually Jacobi forms (as discussed in Section 2.2) since they have a modular and an elliptic transformation property. Having provided a small introduction to the construction of elliptic genera, we turn to a few explicit examples that feature in this thesis. Elliptic genera of Calabi–Yau manifolds of complex dimension d are proportional to Jacobi forms of weight 0 and index $\frac{d}{2}$, as we shall see later.

(1) The elliptic genera of T^{2n}, $n \in \mathbb{Z}_+$

$$\mathbf{EG}_{T^{2n}}(\tau,z) = 0\,. \tag{3.49}$$

(2) The elliptic genus of the $K3$ surface is

$$\begin{aligned}\mathbf{EG}_{K3}(\tau,z) &= 2\phi_{0,1}(\tau,z) \\ &= 8\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right),\end{aligned} \tag{3.50}$$

where $\vartheta_i(\tau,z)$ is the Jacobi theta function (2.44).

The elliptic genus of $K3$ is an index defined on its sigma model. Since $K3$ is a hyper-Kähler manifold, it admits a sigma model description in terms of an $\mathcal{N}=(4,4)$ SCFT. The elliptic genus can be defined for any SCFT with $\mathcal{N}=(2,2)$ or more supersymmetry [**Wit87, Moo07**]. A good physical review of elliptic genera for SCFT's can be found in [**Moo07**]. The physical interpretation of the elliptic genus of $K3$ is that it is the index of $\frac{1}{4}$–BPS states in the sigma model description, and reduces to the Witten index as $\mathbf{EG}_{K3}(\tau,0)$, which is precisely the Euler character. The computation of invariants of Calabi–Yau manifolds is an important problem in enumerative geometry and non–perturbative phenomena in string theory since these indices in many cases capture information of the count of supersymmetric states of a sigma model whose target space is a Calabi–Yau manifold. The appearance of many such enumerative invariants in string theory is therefore no coincidence. For example, topological invariants such as Gromov–Witten invariants, Donaldson–Thomas invariants, Joyce–Song invariants etc. all play a role in understanding the spectra of BPS states in string theory compactified on various Calabi–Yau manifolds. In fact, Gromov–Witten invariants [**Wit91, Vak08**] can be related [**MNOP06**, **MNOP06**] to Donaldson–Thomas [**DT98**]/ Gopakumar–Vafa [**GV98a**, **GV98b**].[3] For more details and references, we refer the reader to [**Kat06**, **KS08**, **DK90**, **JS08**].

[3]Although the author of this thesis has worked on these topics, they will not be discussed here and the reader is referred to [**BCKS20**] for more details.

CHAPTER 4

Finite simple sporadic groups and lattices

4.1. Overview of this chapter

In this chapter, we shall provide some of the basics of group theory, finite group theory, sporadic groups and their representations. Group theory is a core concept in modern mathematics and theoretical physics with a wide range of applications. For a more comprehensive exposition, we refer the reader to [**Wil09**, **Gor13**].

4.2. Group theory basics

Groups are one the most fundamental concepts in mathematics and are relevant for the study of symmetry.

Definition 4.2.1 (Groups). *A group is a set G that is endowed with an operation $\star : G \times G \to G$ satisfying the following conditions:*
Closure: $a \star b = c \ \in G \ \forall \ a, b \in G.$
Associativity: $(a \star b) \star c = a \star (b \star c) \ \forall \ a, b, c \in G.$
Identity element: $\exists \ e \in G$ *such that* $a \star e = e \star a = a.$
Inverse element: $\forall \ a \in G, \ \exists \ a^{-1} \in G$ *such that* $a \star a^{-1} = a^{-1} \star a = e.$

It is not always the case that $a \star b = b \star a$ for generic group elements a, b. However, there are groups for which the group action is commutative and such groups are referred to as *Abelian groups*.
The number of elements of a group G is called the order of G and is denoted by $\mathrm{ord}(G)$. This is not to be confused with the order of an element of G which is the number of times the element must act on itself with respect to the group action to give the identity, e.

Definition 4.2.2 (Subgroups). *A subgroup H of a group G is a subset which maintains a group action i.e., it is itself a group.*

Note that the group action of a subgroup $H \subset G$ is not necessarily the same as the group action of G. All subgroups except for the trivial subgroup $\{e\}$ and G itself are called proper subgroups. A proper subgroup H of G is denoted as $H < G$. A normal subgroup N of G ($N \lhd G$) is a subgroup that is invariant under conjugation with all elements in G i.e.,

$$N \lhd G \Leftrightarrow \ N = \left\{ n \in G \ \middle| \ gng^{-1} = n \ \forall \ g \in G \right\} \ . \tag{4.1}$$

The center $Z(G)$ of a group G is the set of all elements that commute with every other element of the group G i.e.,

$$Z(G) = \left\{ a \in G \;\middle|\; \forall\, g \in G,\ g \star a = a \star g \right\}. \tag{4.2}$$

A closely related object is the centralizer of an element, $C_G(a)$ which is defined as the set of all elements in the group with which it commutes i.e.,

$$C_G(a) = \left\{ g \in G \;\middle|\; a \star g = g \star a \right\} \ . \tag{4.3}$$

Definition 4.2.3 (Conjugacy classes). *Two elements of the group G, $a,\ b \in G$ are conjugate to each other ($a \sim b$) if $\exists\, g \in G$ such that $gag^{-1} = b$. Conjugation has the following properties:*

(1) $a \sim a$.
(2) $a \sim b \;\Leftrightarrow\; b \sim a$.
(3) If $a \sim b,\ b \sim c,$ then $a \sim c$.

This forms an equivalence relation. It allows us to break down G as the disjoint unions of conjugacy classes $[a]$ for $a \in G$ as

$$G = \bigsqcup\, [a],\ [a] = \left\{ b \in G \;\middle|\; gag^{-1} = b,\ g \in G \right\} \ . \tag{4.4}$$

Conjugacy classes, due to the equivalence relation, can be represented by any of its elements. The number of conjugacy classes of a group is its *class number*, $Cl(G)$. All elements in the same conjugacy class have the same order. The notation of the conjugacy classes is of importance: A conjugacy class is always represented as $\mathbb{Z}*$, where $\mathbb{Z}$ is the integer representing the order of the elements of the conjugacy class and $* = A, B, \cdots$.

Definition 4.2.4 (Cosets). *Let $H \subset G$. The left coset of H in G with respect to an element $g \in G$ is the subset $gH = \{gh \mid h \in H\}$. The set of all left cosets of H in G is denoted by G/H. The right coset of H in G with respect to an element $g \in G$ is the set $Hg = \{hg \mid h \in H\}$ and is denoted as $H\backslash G$. The number of left cosets is equal to the number of right cosets and is defined to be the index of H in G.*

A quotient group of G is the set G/N of right cosets which inherits the group structure, where N is a normal subgroup.

4.3. Sporadic groups

4.3.1. Motivating Finite Groups. In physics, often the nature of symmetries of the theory can be explored through the concept of infinite dimensional Lie groups. While Lie groups are of prime importance in physics, their mathematical structure has been known for a long time. For example, the classification and study of Lie groups done over a century ago by Killing [**Kil88**] Cartan [**Car94, Hal15**] and resulted in what is referred to as the *Killing–Cartan classification*. The classification of finite simple groups, however, is a highly non trivial task and was completed only relatively recently [**Con85**]. Finite groups arise in physics when the symmetry of the theory admits only a finite number of morphisms that preserve the structure of the theory. For example, the $\mathbb{Z}_2$ group is a symmetry of certain simple Klein-Gordon models for real scalar fields ϕ:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \lambda\phi^4, \quad \mathbb{Z}_2 : \phi \to -\phi, \quad \mathbb{Z}_2 : \mathcal{L} \to \mathcal{L}, \quad \phi \in \mathbb{R}, \tag{4.5}$$

or for example in the unbroken symmetry phase of the Ising/Heisenberg model, in the classification of crystalline properties of solids etc. In this thesis, we shall consider a special class of finite groups known as sporadic groups. There is one clear connection between sporadic finite groups and physics in the study of moonshine phenomena (see Chapter 5) and its possible connections in being a discrete symmetry group of BPS spectra in string theory [**EOT11, PPV16, GHV12, HPV19**] and conjectured to be related to the symmetry groups of moduli space of Calabi–Yau manifolds, although this connection is not clear yet [**TW15, GHV12, GV12, BCK+18**].

4.4. Finite groups and their classification

Definition 4.4.1 (Finite group). *A finite group is a group G with finite order i.e., ord$(G) < \infty$. A finite simple group is a finite group that is simple i.e., it has no normal subgroup as defined in (4.1).*

If a finite group is not simple, then it can be decomposed into a series of subgroups by quotienting with respect to a normal subgroup. Therefore, in the study of finite groups, finite simple groups (FSGs) are 'atomic'.

Theorem 4.4.1 (Jordan-Hölder). *Let G be a finite group. Let G be decomposable into two chains of subgroups (composition series)*

$$\begin{aligned} \{e\} &= G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G\,, \\ \{e\} &= H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_m = G\,, \end{aligned} \tag{4.6}$$

such that G_{i+1}/G_i and H_{i+1}/H_i are simple and e is the identity element of G. The Jordan-Hölder theorem states that $m = n$ and the two composition series have the same factors up to permutations.

The Jordan-Hölder theorem 4.4.1 implies that any finite group is made up of the same finite simple groups regardless of how they are constructed. This allowed mathematicians to classify the finite simple groups [**Con85**] and this classification scheme is as follows.

Theorem 4.4.2 (Classification of FSGs). *A finite simple group is isomorphic to one of the FSG's of the following type:*

(1) Cyclic Groups ($\mathbb{Z}_p$, p prime).
(2) Alternating groups (A_n, $n > 4$).
(3) 16 families of Lie type.
(4) Sporadic groups, which are further divided into the Pariahs and the Monster family.

The sporadic groups are the only type of finite simple groups that do not form an infinite family.

4.5. Sporadic groups

Sporadic groups come in four classes [**Gri98, Boy13**]: Three classes form consecutive levels (i.e., they can be obtained as systematic quotients/subgroups of the Monster group, $\mathbb{M}$, the largest sporadic group) and a fourth class known as the *Pariahs* (these sporadic groups do not arise from $\mathbb{M}$).

4.5.1. Level 1 sporadic groups: The Mathieu groups. The first level of sporadic groups consist of the five Mathieu groups, $\mathbf{M}_{24}$, $\mathbf{M}_{23}$, $\mathbf{M}_{22}$, $\mathbf{M}_{12}$, $\mathbf{M}_{11}$ with

$$\mathbf{M}_{24} \supset \mathbf{M}_{23} \supset \mathbf{M}_{22} \supset \mathbf{M}_{12} \supset \mathbf{M}_{11}\ . \tag{4.7}$$

The Mathieu groups will be relevant in this thesis since they are the key ingredients in the Mathieu moonshine phenomena.

4.5.2. Level 2 sporadic groups: The Conway groups. The second level of sporadic groups were discovered from the context of sphere packing in that the automorphism group of the Leech lattice is the Conway group, $|\mathrm{Aut}\Lambda_{24}| \equiv Co_0$ [**Lee67, CS13**]. This Co_0 group admits a simple quotient $Co_0/\mathbb{Z}_2 \equiv Co_1$. The other level two sporadic groups are

$$Co_1,\ Co_2,\ Co_3,\ HS,\ McL,\ HJ,\ Suz\ .$$

4.5.3. Level 3 sporadic groups: The Monster groups. The level three sporadic groups were constructed starting with the Monster group, $\mathbb{M}$. The other level 1 sporadic groups are

$$B,\ F_{24},\ F_{23},\ F_{22},\ HN,\ Th,\ He\ .$$

All the level 1, 2, 3 sporadic groups arise as subgroups or quotients of the Monster [**BHL+20**].

4.5.4. Level 4 sporadic groups: The Pariah groups. Finally, we have the Pariah groups which are neither quotients nor subgroups of the Monster group but they do in certain cases admit mappings into other sporadic groups of level 1, 2, 3 [**Gri98, BHL+20**]. These groups are

$$Ru,\ ON,\ Ly,\ J_4,\ J_3,\ J_1\ .$$

For many of the above sporadic groups, there are associated moonshine phenomena. While we will go over moonshine phenomena at a later stage, it suffices to mention briefly what this is. Sporadic groups in many cases are the automorphism groups of a vertex operator algebra and the characters of this algebra in a suitable representations are related to the coefficients of certain (mock) modular objects. Such phenomena have been discovered and studied for many of the above groups, for example: Monster moonshine [**CN79, Bor92**], Conway Moonshine [**Dun05, HKP14**], Baby Monster Moonshine [**Cara, Carb, Carc**], Mathieu moonshine [**EOT11, GPRV13**], Thompson moonshine [**HR16**], O'Nan moonshine [**DMO17**].

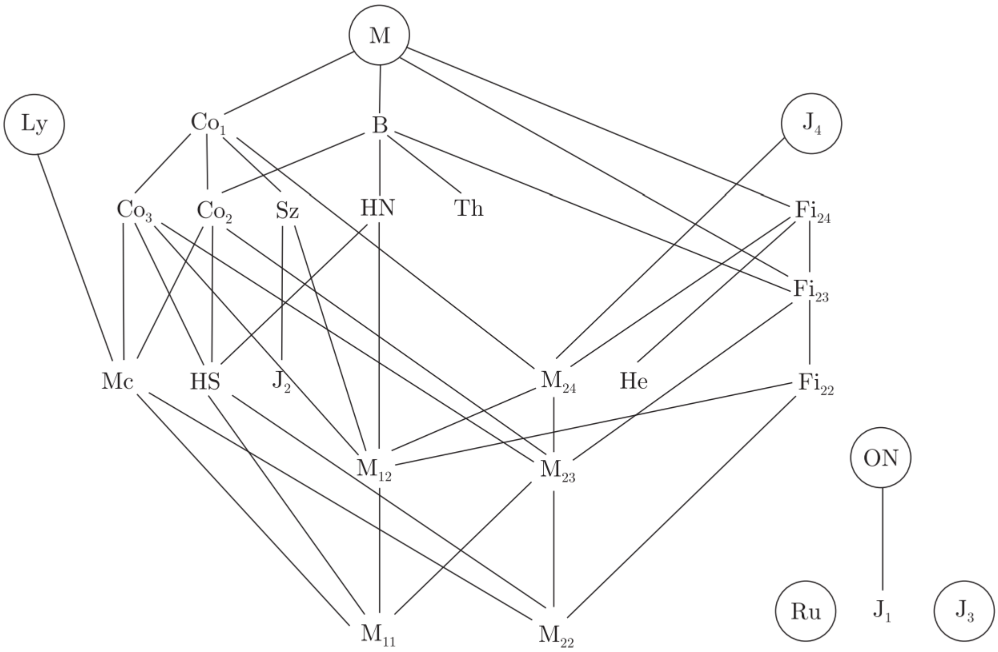


FIGURE 5. Relation between sporadic groups. This picture has been reproduced from [**Boy13**].

4.6. Representations of finite simple groups

In the previous section, we simply listed what the sporadic groups were. Here, we write about the representations of sporadic groups which are key in the understanding of moonshine phenomena. We consider representations over a complex vector space here. Consider a group homomorphism $f : G \to GL(V)$ to the general linear group over V. The vector space V together with the map f, (V, f) is known as the representation of G. The dimension of the representation is obtained from considering the images of $f(g), \quad g \in G$. On $GL(V)$, the images will be an invertible matrix $m \in \mathrm{Mat}_n(\mathbb{C})$ of dimension $n \times n$. In this case, we say that the dimension of the representation is n. The vector space V is called a G–module and admits a G–action. Analogously, for the case of a $\mathbb{Z}_2$ graded vector space $V = V_0 \oplus V_1$ (as in a supersymmetric theory), the $G-$module is a supermodule. The representation is said to be faithful if its kernel is trivial. We shall focus on the case of irreducible representations here. Consider two representations of the kind $f : G \to GL(V)$ i.e., (V, f), (V', f'). The tensor product and the direct sums of their representations lead to new representations $V \oplus V'$, $V \otimes V'$. Two representations are said to be equivalent if $\exists\ m \ \in GL(n, \mathbb{C})$ such that $mf'(g) = f(g)m$, $\forall g \in G$. A subrepresentation of a representation (V, f) is a representation (U, h) where $U \subset V$ that carries the group action of G and $f|_U = h$. A representation is said to be irreducible if it has no subrepresentations. A representation that be be written as a direct sum of finitely many irreducible representations is said to be completely reducible.

THEOREM 4.6.1 (Maschke). *All finite dimensional representations of a finite group may be completely reduced* [**Mas98**].

A dual representation f^* of f is defined as

$$f^*(g) = \left(f(g^{-1})\right)^T, \quad g \in G\,, \tag{4.8}$$

which can be considered to be the representation of G in $V^* = \mathrm{End}(V, \mathbb{C})$.

4.6.1. Character of a representation. A character χ_f of a representation (V, f) is the quantity

$$\chi_f(g) = \mathrm{Tr}(f(g)), \quad g \in G. \tag{4.9}$$

In the case that the vector space is $\mathbb{Z}_2$ graded, the Tr is replaced by $s\mathrm{Tr}(*) = \mathrm{Tr}_{V_0} * -\mathrm{Tr}_{V_1} *$. A character is irreducible if the representation is also irreducible. A character is a class function i.e., $\chi_f(g)$ remains invariant under conjugation with any element $h \sim g$. Two (complex) representations have the same characters iff they are equivalent. Two characters $f,\ f'$ of G form a commutative, associative algebra. If we consider two representations that are unitary then they are equipped with a Hermitian inner product

$$\langle \chi_f, \chi_{f'} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_f(g) \overline{\chi_{f'}(g)}. \tag{4.10}$$

In the case that $(V, f),\ (V', f')$ are irreducible representations for G then two characters $\chi_f,\ \chi_{f'}$ are equivalent if $\langle \chi_f, \chi_{f'} \rangle = 1$. The Hermitian product (4.10) vanishes in the case that the two irreducible representations are not equivalent. The existence of this inner product allows us to interpret these characters as vectors that are orthonormal and span a space. The dimension of this space i.e., the number of irreducible characters that span this space is given by the number of conjugacy classes of G. From (4.10), there exists another orthonormality property for elements in the same conjugacy class as

$$1 = \langle \chi_f(g), \chi_f(h) \rangle = \begin{cases} \dfrac{1}{|C_G(g)|} \displaystyle\sum_f \chi_f(g) \overline{\chi_f(h)}, & h \sim g \\ 0, & \text{otherwise} \end{cases}. \tag{4.11}$$

This means that the number of irreducible representations of a finite group G is equal to the number of conjugacy classes. From (4.11), we can group the $|Cl(G)| \times |Cl(G)|$ characters of G. This is known as the character table. The character tables are usually read off from [**Con85**].

4.7. Lattices and sporadic groups

Just as in the case of Lie theory, there are lattices corresponding to the various sporadic groups. In this section, we review some of the key properties of lattices required for the study of moonshine.

Let W be a vector space on $\mathbb{R}$ such that $\dim W = n < \infty$. Let this vector space be endowed with an inner product $\langle \cdot, \cdot \rangle$. A finite subset $U \subset W,\ U = \left\{ w \in W \,\middle|\, \langle w, w \rangle \neq 0 \right\}$ is called a root system of rank r if

(a) $\mathrm{span}(W) = U$.
(b) The action of reflections on U is closed i.e.,

$$b - 2\frac{\langle a, b \rangle}{\langle a, a \rangle} a \in U \quad \forall\ a,\ b\ \in U. \tag{4.12}$$

This implies that only $a,\ -a\ \in U\ \forall\ a\ \in U$.

(c) $2\frac{\langle a,b\rangle}{\langle a,a\rangle}a \in \mathbb{Z} \ \ \forall\ a,b \in U.$

Elements of a root system are referred to as roots. A root system that cannot be split into proper, orthogonal subsets are said to be irreducible. A root system where all the vectors have the same norm is known as a simply laced root system. For a root system U, the set of roots $\{a_i,\ i=1,\cdots,n\}$ chosen such that $\forall\ b \in U$ with $b=\sum_{i=1}^{n} c_i a_i$ for $c_i \in \mathbb{Z}$ is known as the set of simple roots. The set of simple roots is unique for a root system, up to the action of the group of reflections generated by all roots (i.e., up to $c_i \to -c_i$). This group of reflections is called the Weyl group of U and is denoted by $\mathrm{Weyl}(U)$. As in Lie theory, to every irreducible root system, a connected Dynkin diagram can be associated to it. Each irreducible root system in its set of simple roots contains a highest root $\tilde{a}$ such that

$$\tilde{a} = \sum_{i=1}^{n} c_i a_i \tag{4.13}$$

is such that $\sum_i c_i$ is maximal. The Coxeter number of U is defined as

$$\mathrm{Cox}(U) = 1 + \sum_{i=1}^{n} c_i. \tag{4.14}$$

An even, unimodular, integral, positive lattice Λ of rank n is a free Abelian group isomorphic to $\mathbb{Z}^n$ and is endowed with a symmetric inner product $\langle\cdot,\cdot\rangle_\Lambda$ such that

Positive definiteness: $\langle\cdot,\cdot\rangle_\Lambda$ is positive definite.
Integrality: $\langle x,y\rangle_\Lambda \in \mathbb{Z}\ \forall x,y \in \Lambda$.
Even: $\langle x,x\rangle_\Lambda \in 2\mathbb{Z},\ \ \forall x \in \Lambda$.
Unimodularity: The dual lattice Λ^* is isomorphic to Λ.

The elements of the lattice of norm 2 are known as the roots of the lattice. Lattices play an important role in string compactification and are often used to understand the degrees of freedom in conformal field theories [**GO83**, **LSW89**, **Sch89**]. From the point of viewpoint of moonshine, we shall be interested in even, self-dual, unimodular lattices in 24 dimensions.

Theorem 4.7.1 (Niemeier). *There are only 24 even, self-dual, unimodular lattices in $d = 24$ up to isomorphisms* [**Nie73**].

These lattices consist of the Leech lattice (the only one of the Niemeier lattice with no root vectors) and 23 Niemeier lattices. For each of these 24 lattices, there is an associated finite group

$$G_{\Lambda^{24}} = \mathrm{Aut}(\Lambda^{24})/\mathrm{Weyl}(\Lambda^{24}).$$

For example, for the case of the Leech lattice, the associated group is the Co_0 sporadic group of level 2. Other groups can be obtained by considering sections and/or quotients of these lattices. For a different Niemeier lattice with a different root system, one obtains the $\mathbf{M}_{24}$ group which plays a central role in Mathieu moonshine [**CH15**, **CDA14**, **CDH13**].

Part 2

Moonshine and automorphic forms

CHAPTER 5

Introduction to Moonshine

Overview of this chapter

In this chapter of the thesis, we aim to provide an introduction to the Monster and Mathieu moonshines. The standard references for this subject are [**Gan06**, **AC18**, **Kac16**]. We shall only consider two cases of moonshine viz., the Monster moonshine and Mathieu moonshine phenomena. An introduction to other moonshine phenomena can be found in [**AC18**].

5.1. Drinking up Moonshine

It is seldom the case that a mere numerical equality would lead to deep connections between different areas of mathematics and physics. That is precisely the case with the discovery of moonshine. Often, it is defined as a surprising relation between completely unrelated branches of mathematics viz., number theory (Fourier coefficients of modular forms) and representation theory (dimensions of irreducible representations of sporadic groups). Each of these topics, viz., modular forms and sporadic groups, have been addressed in Chapter 2 and Chapter 4, respectively. The study of moonshine originates with the study of the $j-$ function

$$j(\tau) = \frac{\left(1 + 240 \displaystyle\sum_{i=1}^{\infty} \sigma_3(n) q^n\right)^3}{q \displaystyle\prod_{n=1}^{\infty} (1-q^n)^{24}} = \frac{\Theta_{E_8}(\tau)^3}{\eta(\tau)^{24}}, \tag{5.1}$$

where $\sigma_3(n) = \displaystyle\sum_{d|n} d^3$, $\eta(\tau)$ is the Dedekind–eta function as defined in (2.7a) and Θ_{E_8} is the theta function of the root lattice of E_8.[1] The theta function of a unimodular positive definite, rank n lattice Λ is a modular form of weight $\frac{n}{2}$ on $\mathbb{H}$ defined by

$$\Theta_\Lambda = \sum_{a\in\Lambda} e^{i\pi\tau||a||^2}, \quad \tau \in \mathbb{H}\ , \tag{5.2}$$

such that the Fourier expansion coefficients of the the theta function is the number of vectors of norm $||a||^2$. Since the E_8 root lattice has rank 8, $\Theta_{E_8}(\tau)$ is a modular form of weight 4. This means that the $j(\tau)$-function (5.1) is a weight zero modular form on the upper half plane.[2] John McKay in 1978 noticed that when one considers the

[1] E_8 is a positive-definite, self-dual, even, unimodular lattice.

[2] The $j(\tau)$ is a modular generator for holomorphic modular functions on $\mathbb{H}$ i.e., every holomorphic modular function is a linear function of $j(\tau)$. This means that the $j(\tau)$ is a Hauptmodul on a genus zero Riemann surface.

Fourier expansion of $j(\tau)$, the q–series obtained is

$$j(\tau) = q^{-1} + 744 + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots\ , \tag{5.3}$$

with the odd coincidence that $196884 = 196883 + 1$, where 1 is the dimension of the trivial irreducible representation of the Monster group, $\mathbb{M}$, and 196883 is the dimension of the first non-trivial irreducible representation of $\mathbb{M}$. Before we proceed further we introduce $J(\tau)$ by re-writing $j(\tau)$ as

$$J(\tau) = j(\tau) - 744 = \underset{c_{-1}}{\underset{\parallel}{\mathbf{1}}} \times q^{-1} + \sum_{n\geq 1}^{\infty} \mathbf{c_n} q^n\ . \tag{5.4}$$

Recall from Chapter 2 that topology of the fundamental domain of $SL(2,\mathbb{Z})$ is that of a Riemann sphere. This is the *genus-0* property and is also the characteristic of Fuchsian subgroups of $SL(2,\mathbb{Z})$ [**Har10**]. Since $j(\tau)$ as defined in (5.3) has a Fourier expansion of the form $q^{-1} + C + O(q)$, with C constant, $j(\tau)$ is a *genus–0* function [**Cara**].[3] The observation by McKay was initially checked for more coefficients by Thompson [**Tho79**], and was generalized subsequently by Conway and Norton [**CN79**] where it was demonstrated that the coefficient of the n^{th} term of Fourier expansion of $J(\tau)-$function is a positive linear combination of the dimensions of the irreducible representations of the first n irreducible representations of $\mathbb{M}$,

$$\begin{aligned} 1 &= \mathbf{1} \\ 196884 &= \mathbf{1} + \mathbf{198883} \\ 21493760 &= \mathbf{1} + \mathbf{196883} + \mathbf{21296876} \\ 864299970 &= 2\times\mathbf{1} + 2\times\mathbf{196883} + \mathbf{21296876} + \mathbf{842609326}, \\ &\ \ \vdots \end{aligned} \tag{5.5}$$

where the numbers in bold are the dimensions of the irreducible representations of $\mathbb{M}$. This relation (5.5) was coined *moonshine* [**CN79**]. This particular case of moonshine concerns a modular form of weight zero and is hence a weight zero moonshine. Other weight zero moonshines exist, e.g., the Conway moonshine, while moonshines for half integer weights have also been discovered for the Mathieu, O'Nan, and Thompson groups (c.f. [**AC18**] and references therein). For the course of this thesis, we will be interested in weight $\frac{1}{2}$ moonshine associated with the Mathieu group. However, it serves beneficial to understand the workings of moonshine and we shall hence give an overview of the Monster moonshine before we proceed to the Mathieu moonshine. Over the years, the definition of moonshine, too, has changed from a surprising connection between unrelated fields of mathematics, to a relation between representation theory of finite simple sporadic groups, lattices and vertex operator algebras (VOA's)[4], with the underlying relation stemming from a physical theory (string theory). The nature of the relations is as follows (c.f. Figure 6):

Lattices: There exists a lattice on which a string theory can be studied (Ex: Leech, E_8).

[3]A modular function is said to be genus–0 if it has only one transcendental function as a generator up to $SL(2,Z)$ transforms. Such functions are also called *Hauptmoduln*.

[4]As far as the scope of this thesis is concerned, a VOA can be thought of as being equivalent to a conformal field theory.

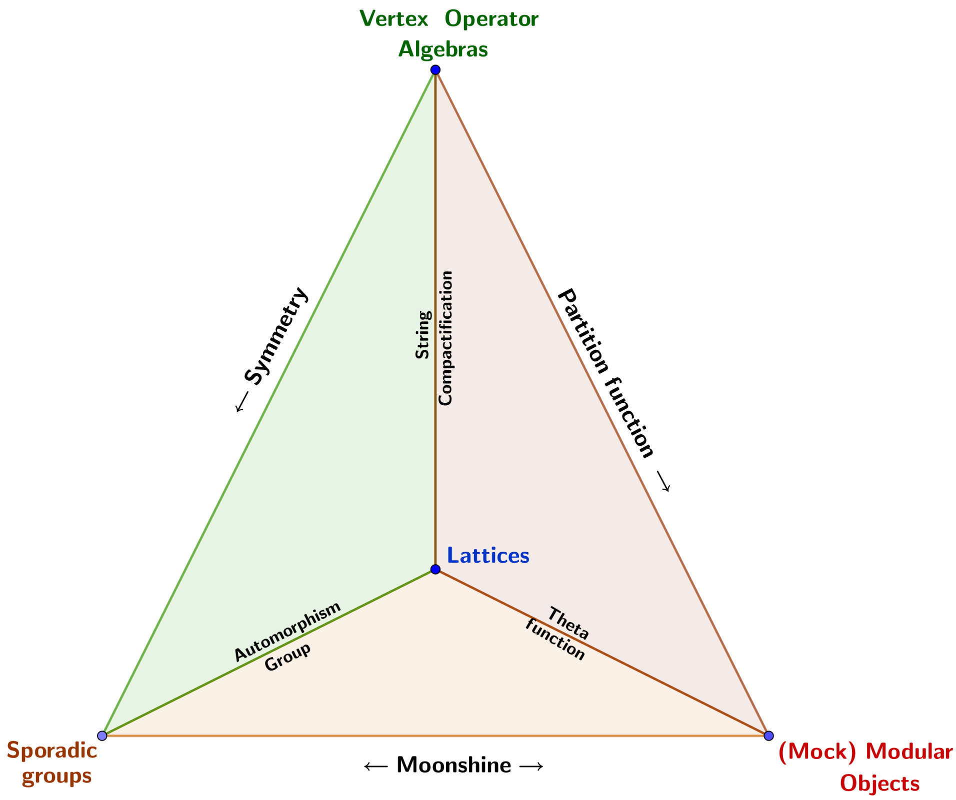


FIGURE 6. The above picture describes the relationship that underlies a moonshine theory. For moonshine to exist, an elaborate structure that relates the sporadic groups, modular objects and VOA's is required.

(Mock) Modular forms: The theta function for this lattice determines the partition function for the string world sheet theory. This partition function is a modular form.

Automorphism groups: An automorphism group of the lattice is a sporadic group (or rather of an algebra defined on the vector space composed of the vectors that span the lattice). For the case of the Monster group, this algebra is a 196883 dimensional algebra, known as the Griess algebra [Gri82].

REMARK 5.1.1. *For example, for the case of the monstrous moonshine: The (orbifolded) Leech lattice, the $J(\tau)$, and the Monster group are related by the VOA which is the holomorphic CFT for 24 free bosons on the Leech lattice with the action of an asymmetric orbifold* [DGH88]. *For the case of Umbral moonshines, the Niemeier lattices, certain vector valued mock-modular forms and the other sporadic groups (of which the $\mathbf{M}_{24}$ is one) get related to each other* [CD12, CDA14, CDH13].

Such relations have been found for many other sporadic groups and their quotients with the relation being manifest due to a physical theory with discrete symmetries (the world sheet theory of a string theory). While it is not entirely obvious if all sporadic groups admit a moonshine type relation, nor if there is a string theory setting for moonshines, there is work in this direction which is promising [BHL+20]. This chapter is an exposition on two moonshine phenomena pertaining to $\mathbb{M}$ and $\mathbf{M}_{24}$.

5.2. An overview of monstrous moonshine

We now continue to provide a self-sustained explanation to (5.5).[5] This observation hints at the existence of a graded representation for the Monster group, $V^\natural$ i.e., these numbers on the L.H.S of (5.5) can be interpreted as the dimensions of vector spaces. This means that

$$V^\natural = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots = V = \bigoplus_{i=-1, i\geq 1}^{\infty} V_i, \tag{5.6}$$

with

$$\begin{aligned} \dim(V_{-1}) &= \chi_0 \\ \dim(V_1) &= \chi_0 \oplus \chi_1 \\ \dim(V_2) &= \chi_0 \oplus \chi_1 \oplus \chi_3 \\ &\cdots, \end{aligned} \tag{5.7}$$

where χ_i are the dimensions of the irreducible representations of $\mathbb{M}$. This implies that (5.1) has the form

$$J(\tau) = \dim_q(V^\natural) = \dim(V_{-1})\ q^{-1} + \sum_{n=1}^{\infty} \dim(V_n) q^n\,. \tag{5.8}$$

Proving the existence of a graded representation satisfying (5.8) is of course trivially possible if one considers multiple copies of the trivial representation of $\mathbb{M}$. A non-trivial proof of requires the so called *McKay–Thompson series.*

DEFINITION 5.2.1 (McKay–Thompson series). *For each element g in $\mathbb{M}$, there exists a series $T_g(\tau) := \sum_{n=-1, n>1} \mathrm{Tr}_{V_n}(g) q^n$ with $T_e(\tau) = J(\tau)$.*

CONJECTURE 5.2.1 (Conway–Norton conjecture on monstrous moonshine). *For each $g \in \mathbb{M}$, T_g is the unique Hauptmodul for a genus-0 subgroup, $\Gamma_g < SL(2,\mathbb{R})$, and this genus-0 subgroup Γ_g contains the congruent subgroup $\Gamma_0(N)$ as a normal subgroup where $ord(g) \times \gcd(24, ord(g))$ is a divisor of N.*

REMARK 5.2.1. *These McKay–Thompson series also satisfy the genus–0 property. Furthermore, these functions are defined with respect to the conjugacy class of the element g i.e., $T_g = T_{h^{-1}gh}$. This means that up to conjugation, there are only a finite number of McKay–Thompson series.*[6] *Furthermore, these functions are defined on a genus–0 subgroup that is not necessarily a subgroup of $SL(2,\mathbb{Z})$ and in general include the Atkin–Lehner involutions which are involutions which identify equivalent cusps but do not constitute an element of a discrete subgroup of $SL(2,\mathbb{Z})$.*

It is not always true that the fundamental domain of a subgroup of $SL(2,\mathbb{Z})$ is topologically genus–0. This makes it non-trivial to provide an interpretation for the modular properties of the McKay–Thompson series. We therefore make a stronger

[5] It is worth noting that while the relation as in (5.5) was observed, it was then not yet certain if the Monster group actually existed.

[6] The Monster has 194 conjugacy classes, and only 171 distinct McKay–Thompson series.

statement on the genus–0 property. The modular properties of the McKay–Thompson series originate from the *Rademacher sum* from the polar term under a subgroup of $SL(2,\mathbb{R})$. The reinterpretation of the genus–0 property as a statement on Rademacher summability will in fact be crucial in the study of Mathieu moonshine. We refer the reader to [**CD14, DF11**] for details regarding Rademacher summability.

5.2.1. Towards a proof of the Conway–Norton conjecture. While we will not prove the Conway–Norton conjecture here, we shall simply give an overview of the proof which will be essential in understanding the proofs of various moonshine conjectures for other sporadic groups. The first step of the proof involves the construction of the Monster module which was constructed in [**FLM89**]. This module, $V^\natural$, is a holomorphic VOA/CFT. This in fact is the CFT of 24 chiral bosons compactified on the Leech lattice. This is a CFT with central charge $c = 24$. Since the eigenvalue of the L_0 Virasoro operator is 0, the lowest term in the q–series of this partition function is indeed q^{-1}, as is required for the genus–0 property/Rademacher summability. The partition function of this physical set up is in fact related to the $J(\tau)$ function (5.4) as

$$Z_{V^\natural}(\tau) = J(\tau) + 24 = \frac{\Theta(\Lambda_{Leech})}{\eta(\tau)^{24}}. \tag{5.9}$$

The additional 24 fields which form weight 1 primaries can be removed by performing an asymmetric $\mathbb{Z}_2$ orbifold which gives the partition function as being precisely $J(\tau)$ as in (5.4).[7] Under this orbifold, the automorphism group reduces to a discrete subgroup of the Monster group, Co_0, up to a multiplicative factor. Since the orbifold removes the weight 1 states, in looking at the weight 2 states, the interpretation from the q–series (5.3) is that the weight 2 states form a 196883 dimensional space. The algebra over this space is a commutative, non-associative algebra known as the *Griess algebra* [**Gri82**]. This Griess algebra has an automorphism group which is the Monster group. Therefore, we may conclude that $\mathbb{M}$ is indeed the automorphism group of $V^\natural$.

5.2.2. Borcherds' proof of the Conway–Norton conjecture. The remaining part of understanding the proof of the monstrous moonshine conjecture is to show that the modules constructed in [**FLM89**] are precisely the Hauptmoduln of subgroups of the Monster. In other words,

$$T_g^{V^\natural}(\tau) = \mathrm{Tr}_{V^\natural}\left(g\ q^{L_0 - \frac{c}{24}}\right) \tag{5.10}$$

are the Hauptmoduln for Γ_g. Simply put, this means that the coefficients of the Fourier expansion of $T_g^{V^\natural}(\tau)$ using the Rademacher technique should agree with the coefficients of the dimensions of the Hauptmoduln. For the latter, one may use the Koike–Norton–Zagier [**GZ84**] identity which is the product identity that takes the $J(\tau)$ function and allows for the exact computation of the Fourier expansion coefficients, c,

$$p^{-1} \prod_{\substack{m>0 \\ n\in\mathbb{Z}}} (1 - p^m q^n)^{c(mn)} = J(\sigma) - J(\tau), \tag{5.11}$$

[7]This is in fact the first construction of an asymmetric orbifold, historically speaking.

where $c(mn)$ is the mn^{th} coefficient in the Fourier expansion of $J(\tau)$.[8] This product identity is sufficient to constrain the all coefficients from a few (five) low lying coefficients i.e., $c(1), \cdots, c(5)$. Using this formula, a generalized Kac–Moody algebra which is a Kac–Moody algebra with imaginary roots was constructed [**Bor92**] . It was proven that the algebra is precisely the Monster Lie algebra, $\mathfrak{m}$. Using the denominator identities from the Weyl–Kac character formula for a Lie algebra, a Weyl denominator formula which is related to the coefficients of $J(\tau)$ was constructed [**Bor92**]. Subsequently, by considering the action of $g \in \mathbb{M}$ on this denominator identity, it was shows that the twisted/twined denominator identities are related to the product identity of the $J-$invariants defined on Γ_g, i.e., the Hauptmoduln [**Bor92**]. In doing so, it was shown that the construction of the VOA in [**FLM89**] is indeed the VOA required to prove the Conway–Norton moonshine conjecture.

5.2.3. Generalized Monstrous Moonshine. We now turn to the concept of generalized monstrous moonshine as was first proposed in [**Nor87**] and subsequently studied in [**Cara, Carb**] and proven in [**Carc**]. The idea behind generalized moonshine is to assign to each element $g \in \mathbb{M}$ a graded projective representation of the centralizer group of $\mathbb{M}$, $C_{\mathbb{M}}(g)$

$$V(g) = \bigoplus_{n \in \mathbb{Q}} V(g)_n \ , \tag{5.12}$$

and to each commuting pair of elements $(g, h) \in \mathbb{M}$, $\exists\ T_{(g,h)}$ that is a holomorphic function on $\mathbb{H}$. $T_{(g,h)}$ satisfies the following conditions

(a) $T_{g^a h^c, g^b h^d}(\tau) = \gamma\, T_{(g,h)}\left(\frac{a\tau + b}{c\tau + d}\right)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and γ is a 24^{th} root of unity.
(b) $T_{(g,h)}(\tau) = T_{k^{-1}gk, k^{-1}hk}(\tau)$, $k \in \mathbb{M}$.
(c) $T_{(g,h)}(\tau) = \sum_{n \in \mathbb{Q}} \mathrm{Tr}_{V(g)_n}\left(\tilde{h} q^{n-1}\right)$, where $\tilde{h}$ is a lift of h.
(d) $T_{(g,h)}(\tau)$ is one of either a constant or a Hauptmodul for some genus zero CSG $\Gamma \subset SL(2, \mathbb{Z})$.
(e) $T_{(g=e,h)}(\tau)$ reduces to the McKay–Thompson series for some element $h \in \mathbb{M}$.

These functions $T_{(g,h)}$ can be interpreted as twisted–twined characters of $V^\natural$ defined in (5.6) [**DGH88**].

5.2.4. String theory and monstrous moonshine. While it is a remarkable relation, the true interpretation of the monstrous moonshine in physics is still a matter needing resolution. One of the first possible interpretations of the moonshine phenomenon was studied in [**Wit07**] where a suggestion was that the Monster group could be the symmetry of an infinite family of two dimensional conformal field theories with central charge $c = 24k$ dual to AdS_3 gravity at different values of cosmological constants. A good review on partition functions of AdS_3 gravity is [**Kra08**]. For most of these central charges, such a relation has been ruled out [**Höh08, Gai12, Gab07**]. Subsequent work in [**MW10**] suggested that some of the number theoretic properties required for the construction of such monstrous functions are not favourable from AdS_3

[8] This product formula is also manifest in BPS algebras and hence, moonshine phenomena have been linked to the study of BPS algebras [**HM96, HM98, Gov11, GS19, GGK10, GHP11, PPV17**].

gravity. However, it has been suggested that chiral gravity theories [**Man07, LSS08**] could be dual to the holomorphic part of the Monster CFT. This suggestion has also been studied in [**DF11**] where the $2+1$ dimensional partition function of AdS_3 is constructed by computations of the McKay–Thompson series using the Rademacher summation techniques.
Another relation, motivated by the fact that the moonshine module can also have a $\mathbb{Z}_2$ grading [**DGH88**], is to study the partition function as a partition function of BPS states. This was done in [**PPV16, PPV17**] in considering heterotic quantum mechanics i.e., certain compactifications of all the spatial dimensions of heterotic string theory down to $0+1$ dimensions. The vector space decomposition as in (5.6) is endowed with the interpretation of being the decomposition of the BPS Fock space into subspaces, and the generalized McKay–Thompson series $T_{(g,h)}$ as being twisted–twined BPS partition functions.

5.3. Mathieu Moonshine

While in the previous section we introduced the theory of moonshine and how one would go about proving a necessary moonshine phenomenon, we shall now begin with the study of the Mathieu moonshine associated with the $\mathbf{M}_{24}$ group. Unlike the monstrous moonshine, the Mathieu moonshine is a moonshine associated to weight a $\frac{1}{2}$ mock modular form. Mathieu moonshine is of considerable interest to string theorists due to its relation with the moduli space of $K3$ surfaces. In this section, we shall go over the fundamentals of Mathieu moonshine and connections to mock modular forms.

5.3.1. Mathieu moonshine and the elliptic genus of $K3$. The central object in the study of Mathieu moonshine is the elliptic genus of the $K3$ surface that was introduced in Chapter 2 and Chapter 3. The elliptic genus in this case is an index that counts BPS states in the sigma model of $K3$ and is independent of moduli of all the theory.[9] The non–linear sigma model (NLSM) associated to a $K3$ surface is an $\mathcal{N}=(4,4)$ SCFT with central charge $c=\bar{c}=6$. The elliptic genus is expressed as

$$\mathbf{EG}_{K3}(\tau,z)=\mathrm{Tr}_{RR}\left((-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}y^{J_0}\right), \tag{5.13}$$

where $L_0,\ \bar{L}_0$ are the holomorphic and anti–holomorphic Virasoro generators and J_0 is the third component of the $SU(2)$ $R-$symmetry generator that rotates the supercharges into each other, and $-(1)^F=(-1)^{F_L+F_R}$ is the fermion number operator. The trace is taken over the Ramond–Ramond Hilbert space although this may also defined over the NS–NS Hilbert space due to spectral flow symmetry. The elliptic genus is in fact (up to a numerical factor) a weakly holomorphic Jacobi form of weight 0 and index 1 on $SL(2,\mathbb{Z})$ (see Section 2.2),

$$\mathbf{EG}_{K3}(\tau,z)=2\ \phi_{0,1}(\tau,z)=8\sum_{i=2}^{4}\left(\frac{\vartheta_i(\tau,z)^2}{\vartheta_i(\tau,0)^2}\right), \tag{5.14}$$

where $\vartheta_i(\tau,z)$'s are the Jacobi theta functions defined in (2.44). The elliptic genus (5.14) is a topological invariant over the $K3$ moduli space. Due to the hyper–Kähler structure of $K3$, the holonomy allows us to extend the supersymmetry of the worldsheet

[9]For a continuous spectrum, holomorphicity is not necessarily guaranteed. In fact the elliptic genus behaves like a mock modular form [**Tro10**]. In many cases certain BPS indices exhibit wall crossing, i.e., the value of the indices jump as one moves around in the moduli/parameter space of the theory [**Cec10, KS08**].

from $\mathcal{N} = 2 \to \mathcal{N} = 4$ and therefore the Hilbert space of the RR sector may be expressed in terms of representations of the $\mathcal{N} = (4,4)$ SCFT at $c = \bar{c} = 6$. This decomposition is as follows:

$$\mathbf{EG}_{K3}(\tau, z) = 24\,\mathbf{ch}^{\mathcal{N}=4}_{6,0,0}(\tau, z) + \sum_{n=0}^{\infty} A^{(1)}_n \mathbf{ch}^{\mathcal{N}=4}_{6,n,\frac{1}{2}}(\tau, z), \tag{5.15}$$

where the $\mathbf{ch}^{\mathcal{N}=4}_{*}(\tau, z)$ are the characters of the $\mathcal{N} = 4$ superconformal algebra described in Appendix A. The observation in [**EOT11**] is that in the $\mathcal{N} = 4$ decomposition of (5.15), the coefficients are related to the dimensions of the irreducible representations of the $\mathbf{M}_{24}$ group as $24 = 23 + 1$ and other coefficients $A^{(1)}_n$ given by[10]

$$\begin{aligned} A^{(1)}_0 &= -2 = -1 - 1, \\ A^{(1)}_1 &= 90 = 45 + \overline{45}, \\ A^{(1)}_2 &= 462 = 231 + \overline{231}, \\ A^{(1)}_3 &= 1540 = 770 + \overline{770}, \cdots, \end{aligned} \tag{5.16}$$

where $45, 231, 770, \cdots$ are the dimensions of the irreducible representations of $\mathbf{M}_{24}$. This relationship between the elliptic genus of $K3$ represented using characters of the $\mathcal{N} = (4,4)$ sigma model SCFT, and the dimensions of irreducible representations of $\mathbf{M}_{24}$ is known as *Mathieu moonshine*.

5.3.2. Mock-modularity and Mathieu moonshine. Unlike the case of the monstrous moonshine discussed in Section 5.2, the Mathieu moonshine is a moonshine phenomenon associated to a mock–modular form that has been discussed in (2.35) in Section 2.4. The elliptic genus of $K3$ can be re-written as

$$\mathbf{EG}_{K3}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^3}\left(24\mu(\tau, z) + H(\tau)\right), \tag{5.17}$$

where

$$\mu(\tau, z) = \frac{-i\sqrt{y}}{\vartheta_1(\tau, z)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n y^n q^{n(n+1)/2}}{1 - yq^n}, \tag{5.18}$$

and the mock modular form $H(\tau)$ as studied in (2.35) has the Fourier decomposition in terms of

$$H(\tau) = 2q^{-1/8}\left(-1 + 45\;q + 231\;q^2 + 770\;q^3 + \cdots\right). \tag{5.19}$$

The mathematical interpretation of such a split of $\mathbf{EG}_{K3}(\tau, z)$ is surplus to the scope of this thesis but can be found in §8.1 of [**AC18**]. Analogous to the case of the Monster moonshine, there is an infinite dimensional $\mathbb{Z}$–graded module $W = \bigoplus_{n=1}^{\infty} W_n$ for $H(\tau)$

[10]Note that the character $\mathbf{ch}_{6,0,\frac{1}{2}}(q, y)$ here is in a massive representation which can be written as a sum of two massless representations, $\lim_{h \searrow \frac{1}{4}} \mathbf{ch}^{\mathcal{N}=4}_{6,h-\frac{1}{4},l}(q, y) = \mathbf{ch}^{\mathcal{N}=4}_{6,0,\frac{1}{2}}(q, y) + 2\mathbf{ch}^{\mathcal{N}=4}_{6,0,0}(q, y)$. Furthermore, the correct decomposition of the higher coefficients can be fixed by requiring the twined elliptic genera to behave appropriately, see (5.22) and the following discussion.

such that

$$H(\tau) = q^{-1/8}\left(-2 + \sum_{n=1}^{\infty} \dim(W_n)\, q^n\right). \tag{5.20}$$

The existence of such a module W has been studied and proven to exist [Gan16].

5.3.3. On symplectomorphisms of $K3$, and symmetry surfing. Although the algebraic structure of Mathieu moonshine has been studied well [Gan16], its relation to the $K3$ elliptic genus is still unclear. The first hint of a relation between the $K3$ surface and the Mathieu sporadic group is the following theorem [Muk88, Kon98].

THEOREM 5.3.1 (Kondo). *Any finite group whose action acts as an automorphism that preserves the symplectic form of a $K3$ surface can be embedded into $\mathbf{M}_{23} \subset \mathbf{M}_{24}$.*[a]

[a]An automorphism that preserves the symplectic form is a symplectomorphism, for lack of a shorter term.

The question is as to what mathematical object related to the $K3$ surface has the $\mathbf{M}_{24}$ as a symmetry group.[11] The natural candidate is to consider the world sheet NLSM with $K3$ as target space. An excellent exposition on the study of $K3$ NLSM's is [NW01].

THEOREM 5.3.2 (Gaberdiel–Hohenegger–Volpato). *The group of symplectomorphisms of any $K3$ surface is never $\mathbf{M}_{24}$, but is rather necessarily a subgroup of Co_1* [GHV12].

To circumvent this theorem, we turn to the idea of symmetry surfing [TW13a, TW15]. The idea of symmetry surfing is that since the elliptic genus of $K3$ surfaces is a count of $\frac{1}{4}$–BPS states, $\mathbf{M}_{24}$ is necessarily a symmetry of only such BPS states in the $\mathcal{N} = (4,4)$ NLSM of $K3$ surfaces. At any point in the moduli space of the $\mathcal{N} = (4,4)$ sigma model of $K3$, the elliptic genus in the $\mathcal{N} = (4,4)$ representations is expected to correspond to the graded dimensions of some subspace of BPS states of the theory.[12] The idea of symmetry surfing is therefore that the combined set of generators of the symmetries of sigma models of the $K3$ results in the full $\mathbf{M}_{24}$ group. A study by [GTVW14] shows a $K3$ NLSM with symmetry $\mathbb{Z}_2^8 : \mathbf{M}_{20}$, which is one of the largest subgroups of $\mathbf{M}_{24}$.

REMARK 5.3.1. *It also remained unclear if whether the $\mathbf{M}_{24}$ is a symmetry of some object related to $K3$ surfaces, or if it is simply a property of the Jacobi form $\phi_{0,1}(\tau, z)$ that appears in the elliptic genus of $K3$. This question has been addressed in* [BCK+18]*, and will be the focus of Chapter 6.*

Progress towards a complete understanding of Mathieu moonshine has been made in [GHV12, CH15, DMC15, CHVZ16, PVZ17], through the concept of symmetry surfing [TW13b, TW15, GKP17] and other directions [KPV17].

[11]We also refer the reader to [Muk20] for more on the relation between $K3$ and $\mathbf{M}_{24}$

[12]The symmetry of the $K3$ sigma model is usually trivial at generic points in the moduli space.

5.3.4. Generalized Mathieu Moonshine. Similar to the case of generalized Monster moonshine [Cara, Carb, Carc] there exists an infinite dimensional graded module [Gan16]

$$\mathcal{H}^{BPS} = \bigoplus_{n=0}^{\infty} H_n \otimes \mathcal{H}_n^{\mathcal{N}=4}, \tag{5.21}$$

where the sum runs over the contributing irreducible $\mathcal{N} = (4,4)$ representations $\mathcal{H}_n^{\mathcal{N}=4}$ and H_n are $\mathbf{M}_{24}$ representations (in general not irreducible) with $\dim(H_n) = |A_n^{(1)}|$ as in (5.20) [GHV10a]. This implies that the twined elliptic genera of $K3$ obtained by inserting an element $g \in \mathbf{M}_{24}$ in (5.13), i.e.,

$$\mathbf{EG}_{K3,g}(\tau, z) = \mathrm{Tr}_{RR}\left(g\,(-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}y^{J_0}\right), \tag{5.22}$$

form an analogue of the McKay–Thompson series, i.e., they admit an expansion similar to (5.15) but now with coefficients $\tilde{A}_n = Tr_{H_n}(g)$ and these twined elliptic genera (5.22) transform as a Jacobi form of weight 0 and index 1 under the subgroup $\Gamma_0(N)$ of $SL(2,\mathbb{Z})$. Since not all these subgroups are Fuchsian, the analogues of the McKay–Thompson series are not genus–0 functions, but are Rademacher summable instead [CD14]. Based on these facts, in [Che10, GHV10b, GHV10a, EH11] explicit expressions for all the twined genera have been found. For some cases where $g \in \mathbf{M}_{23}$ admits an interpretation as a symplectic automorphism of $K3$ the corresponding twined genera have been calculated directly in [DJS06]. The symmetries of NLSM on $K3$ have been classified in [GHV12, CHVZ16]. Furthermore, all possible twining genera of NLSM on $K3$ have been conjectured in [CHVZ16] based on works of [CH15, DMC15]. This conjecture has been proven in a "physical" way by demanding absence of unphysical wall–crossings in [PVZ17].

Analogous to generalized moonshine, in [GPRV13] all the twisted (by g), twined (by h) elliptic genera $\mathbf{EG}_{K3,g,h}$ have been calculated. For every commuting pair of $(g,h) \in \mathbf{M}_{24}$, the twisted–twined elliptic genera are defined as

$$\mathbf{EG}_{K3,g,h}(\tau, z) = \mathrm{Tr}_{RR,g}\left(h\,(-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{L_0-\frac{\bar{c}}{24}}y^{J_0}\right), \tag{5.23}$$

where the trace is now taken over the g–twisted Ramond–Ramond sector. The twisted–twined elliptic genera (5.23) are expected to fulfill certain properties. They have the following modular and elliptic transformation properties:

(5.24)

$$\mathbf{EG}_{K3,g,h}(\tau, z+\ell\tau+p) = e^{-2\pi i m(\ell^2\tau+2\ell z)}\mathbf{EG}_{K3,g,h}(\tau, z),\ \ \ell, p \in \mathbb{Z}$$

$$\mathbf{EG}_{K3,g,h}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \chi_{g,h}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)e^{2\pi i\frac{cz^2}{c\tau+d}}\mathbf{EG}_{K3,h^cg^a,h^dg^b}(\tau, z),\ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}),$$

for a certain multiplier $\chi_{g,h} : SL(2,\mathbb{Z}) \to U(1)$. In the above equation (5.24), m is the index of the elliptic genus. The multipliers are assumed to be constant under conjugation of the pair (g,h) by an element of $k \in \mathbf{M}_{24}$up to a phase $\xi_{g.h}(k)$ (that depend on certain 2–cocycles c_g) and to have a well defined expansion in terms of $\mathcal{N} =$ 4 superconformal characters. In particular, $\mathbf{EG}_{K3,e,g}$ agrees with the corresponding twined character (5.22). It was postulated in [GPRV13] that these properties (in particular $\chi_{g,h}$, $\xi_{g,h}$ and c_g) are all controlled by a 3–cocycle α representing a class in $H^3(\mathbf{M}_{24}, U(1))$. Moreover, the twisted, twined and twisted–twined genera transform among each other following modular transformations as in (5.24). The set of twisted–twined elliptic genera modulo modular transformations is in one to one correspondence with conjugacy classes of Abelian subgroups of $\mathbf{M}_{24}$ generated by two elements (g, h).[13] While calculating these genera using the properties mentioned above, many of them vanish identically owing to obstructions, i.e., situations where the properties only allow for vanishing characters. Using the shorthand $\mathbf{EG}_{K3,g,h} := \phi_{g,h}$, the only un-obstructed twisted–twined elliptic genera (excluding twined genera) are

$$\phi_{2B,4A_2}, \phi_{4B,4A_3}, \phi_{4B,4A_4}, \phi_{2B,8A_{1,2}}, \underbrace{\phi_{3A,3A_3}}_{=0}, \underbrace{\phi_{3A,3B_1}}_{=0}, \tag{5.25}$$

where we use the short hand that g is an element of the first conjugacy class in the subscript and h is an element of the second conjugacy class in the subscript. For example, $\mathbf{EG}_{K3,g,h} = \phi_{2B,4A_2}$ implies that $g \in 2B$ and $h \in 4A_2$. The conjugacy classes are listed in Appendix B.

5.3.5. String theory and Mathieu moonshine. In this final section, we wish to emphasize more on the connections between string theory and Mathieu moonshine. While there are generalizations of the Mathieu moonshine (*Umbral moonshine*) [**CDA14, CDH13, DGO15, CH15, KPV17**] which have connections to string theory, we shall not review them here for not having reviewed umbral moonshine. The string theoretic origin of Mathieu moonshine has been studied from various points of view. In the previous sections, we have reviewed the $K3$ string theory aspects of Mathieu moonshine. Mathieu moonshine has also been studied in the context of $\mathcal{N} = 2$ supersymmetric compactifications [**CD17a, CD18, CD17b, BCKS20, BCK$^+$20**]. Mathieu moonshine has also been studied from the point of view of NS5 branes in type IIB string theory [**HM14, HMN15**]. Mathieu moonshine, in particular the NSLM studied in [**GTVW14**] has also been studied from the point of view of quantum error correction codes [**HM20**].

The author of this thesis has extensively worked on the following two key questions:

(a) Is the Mathieu moonshine really a property of $K3$ surfaces or is it a property of the Jacobi form $\phi_{0,1}(\tau, z)$? This will be reviewed in Chapter 6.

(b) If the Mathieu moonshine is a property of the BPS states in the sigma model, does this moonshine property also show up in other BPS counting functions, such as Gromov–Witten invariants [**BCKS20**]? The results of this publication will not be reviewed in this thesis.

[13]There are 55 such subgroups, 21 of which correspond to cyclic subgroups generated by elements of the form (e, g) and are associated to twining genera $\mathbf{EG}_{K3,g}$ [**GHV10b**]. This includes the twisted and twisted–twined genera obtained from the twined genera by modular transformations. The remaining 34 have been calculated in [**GPRV13**] using the properties discussed above.

CHAPTER 6

Moonshine in the moduli space of higher dimensional Calabi–Yau manifolds: A computational approach

Overview of this chapter

This chapter is based on results published in [**BCK[+]18**].

In the previous chapter, we introduced the concept of moonshine and in particular afforded a short introduction to Mathieu moonshine in Section 5.3. Some of the closing arguments from the previous chapter 5 were that

(a) The exact nature of the relation between the $K3$ surface elliptic genus, $\mathbf{EG}_{K3}(\tau, z)$ as defined in (5.14) and the $\mathbf{M}_{24}$ group is unclear. This stems from the work of [**GHV12, TW13a, TW15, GKP17, GTVW14**].

(b) Even more basic, it is still unclear if whether the Mathieu moonshine phenomenon is associated to the Jacobi form that represents the elliptic genus of $K3$, or rather something deep associated to $K3$ surfaces.

In this chapter, we analyze the second question. The way we do so is by studying the elliptic genus of a CY_5. The reasons for this are as follows:

(a) When the elliptic genus of CY_5 is expanded in the characters of the $\mathcal{N} = (2, 2)$ NLSM, it exhibits almost the exact same expansion coefficients as the case with the $\mathbf{EG}_{K3}(\tau, z)$ expanded in $\mathcal{N} = 4$ characters.

(b) The elliptic genus of CY_5 can be computed in a straightforward way since it is a Jacobi form of weight 0 and index $\frac{5}{2}$ [**KYY94**].[1] As we shall see later, the elliptic genus of a CY_5 is related to the elliptic genus of $K3$. This poses a nice setting to therefore investigate the role of $\phi_{0,1}(\tau, z)$ in Mathieu moonshine.

The key idea here is to compute the elliptic genus for a large number of CY_5 and study their expansion in terms of the $\mathcal{N} = 2$ characters. These characters have been detailed in Appendix A. The key question here is, "What is the interpretation if one finds interesting expansion coefficients (moonshine phenomena) in higher dimensional manifolds?" If the expansion coefficients are given in terms of irreducible representations of a particular sporadic group, does this imply that all manifolds with such elliptic genera are connected to the particular sporadic group, or only a few, or none? How would such a connection manifest itself? As a geometric symmetry of the manifold or as a symmetry of the non-linear sigma model with the manifold as target space or via something else like symmetry surfing?

In this chapter, we aim to see if whether the elliptic genus of CY_5 carries the action of $\mathbf{M}_{24}$.[2] By calculating twined elliptic genera for more than ten thousand

[1]The Eichler–Zagier theory of Jacobi forms deals only with Jacobi forms of integer index ≥ 1. However, this can been extended to weakly holomorphic Jacobi forms of half–integer index (cf. Lemma 1 in [**Gri99**]).

[2]The case of $d = 4$ is still work in progress [**Kid**].

explicit examples, we investigate the possibility of these CY_5 manifolds being related to Mathieu or Enriques moonshine. There is no systematic construction of $\mathrm{CY}_{d>5}$; we therefore refrain from a detailed investigation thereof. The chapter is organized as follows: In Section 6.1 we review some properties of Jacobi forms used to construct the elliptic genera of various Calabi–Yau manifolds, and their decomposition in terms of $\mathcal{N} = 2$ superconformal characters. We discuss the elliptic genus of Calabi–Yau manifolds with different dimensions as well as its expansion in $\mathcal{N} = 2$ characters. In Section 6.3 we calculate twined elliptic genera for a large number of CY_5 manifolds. We comment on a simple toroidal orbifold and two Gepner models in Section 6.5. We summarize our findings in Section 6.6.

6.1. Jacobi forms and Elliptic Genera of Calabi–Yau Manifolds

We have already encountered the theory of Jacobi forms in Section 6.1 and the construction of elliptic genera in Chapter 3, and the fact that the elliptic genera of CY_d manifolds are weakly holomorphic Jacobi forms of weight 0 and index $\frac{d}{2}$ [**KYY94, Gri99**]. A useful fact to recall here is that the bigraded ring of weakly holomorphic Jacobi forms is generated by only a few elements viz., $\phi_{0,1}(\tau,z)$, $\phi_{-2,1}(\tau,z)$, $E_4(\tau)$, $E_6(\tau)$. This means that the elliptic genera of Calabi–Yau manifolds too are linear functionals of these modular and Jacobi forms.

$$
\begin{aligned}
&\mathbf{EG}_{CY_2} \xrightarrow{\text{Generated by}} \tilde{J}^{!}_{0,1} = \langle \phi_{0,1} \rangle,\\
&\mathbf{EG}_{CY_4} \xrightarrow{\text{Generated by}} \tilde{J}^{!}_{0,2} = \langle \phi_{0,1}^2, E_4\,\phi_{-2,1}^2 \rangle,\\
&\mathbf{EG}_{CY_6} \xrightarrow{\text{Generated by}} \tilde{J}^{!}_{0,3} = \langle \phi_{0,1}^3, E_4\,\phi_{-2,1}^2\,\phi_{0,1}, E_6\,\phi_{-2,1}^3 \rangle,\\
&\mathbf{EG}_{CY_8} \xrightarrow{\text{Generated by}} \tilde{J}^{!}_{0,4} = \langle \phi_{0,1}^4, E_4\,\phi_{-2,1}^2\,\phi_{0,1}^2, E_6\,\phi_{-2,1}^3\,\phi_{0,1}, E_4^2\,\phi_{-2,1}^4 \rangle,\\
&\mathbf{EG}_{CY_{10}} \xrightarrow{\text{Generated by}} \tilde{J}^{!}_{0,5} = \langle \phi_{0,1}^5, E_4\,\phi_{-2,1}^2\,\phi_{0,1}^3, E_6\,\phi_{-2,1}^3\,\phi_{0,1}^2, E_4^2\,\phi_{-2,1}^4\,\phi_{0,1}, E_4\,E_6\,\phi_{-2,1}^5 \rangle.
\end{aligned}
\tag{6.1}
$$

The functions above appear in the elliptic genus of Calabi–Yau $d = 2,4,6,8,10$ manifolds and their coefficients can be fixed in terms of a few topological numbers of the CY_d manifold viz., their Hodge numbers.

For weight zero Jacobi forms with half integer index one may use the map between Jacobi forms of even weight and integer index to relate them to Jacobi forms of even weight and half–integer index [**Gri99**] as follows

$$
J_{2k,m+\frac{1}{2}} = \phi_{0,\frac{3}{2}}\, J_{2k,m-1}\,, \qquad m \in \mathbb{Z}\,, \tag{6.2}
$$

where

$$
\begin{aligned}
\phi_{0,\frac{3}{2}}(\tau,z) &= 8\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau,0)}\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)},\\
\phi_{0,\frac{5}{2}}(\tau,z) &= \phi_{0,1}(\tau,z)\phi_{0,\frac{3}{2}}(\tau,z)\,,
\end{aligned}
\tag{6.3}
$$

with the Jacobi theta functions as defined in (2.44). In particular $\phi_{0,\frac{3}{2}}$ and $\phi_{0,\frac{3}{2}}\phi_{0,1}$ are, up to re–scaling, the unique Jacobi forms of weight 0 and index $\frac{3}{2}$ and $\frac{5}{2}$, respectively. More generally, the space $J_{0,m+\frac{3}{2}}$ is spanned by m functions for $m = 1,2,3,4,5$ and these functions are the ones given in equation (6.1) multiplied by $\phi_{0,\frac{3}{2}}$.[3] We therefore

[3]Note that we have the identity $432\phi_{0,\frac{3}{2}}^2 = \phi_{0,1}^3 - 3E_4\,\phi_{-2,1}^2\,\phi_{0,1} - 2E_6\,\phi_{-2,1}^3$.

see that the space of Jacobi forms $J_{0,\frac{d}{2}}$ is generated by a small bigraded ring for small d. This decomposition (6.2) extends to weakly holomorphic Jacobi forms too and therefore the elliptic genera for CY_d, $d \in 2\mathbb{Z}+1$ may also be easily determined. This means that the elliptic genus of a higher dimensional Calabi–Yau manifold can be fixed in terms of the elliptic genus of a lower dimensional one.

6.2. Elliptic genera of Calabi–Yau's in superconformal character representation

6.2.1. Calabi–Yau 1–folds. For the torus T^2, the elliptic genus vanishes i.e.,

$$\mathbf{EG}_{T^2}(\tau, z) = 0\,. \tag{6.4}$$

The same holds true for any even dimensional torus $\mathbf{EG}_{T^{2n}}(\tau, z) = 0$, $\forall n \in \mathbb{N}$. This is due to the fermionic zero modes in the right moving Ramond sector:

$$\mathrm{Tr}\left((-1)^{F_R}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) = 0\,. \tag{6.5}$$

It must however be pointed that there are constructions for BPS indices in string compactifications of T^6 which are calculable despite the vanishing of the $\mathbf{EG}_{T^6}(\tau, z)$ [**MMS99**].

6.2.2. Calabi–Yau 2–folds. At complex dimension 2, there are two distinct Calabi–Yau manifolds viz., T^4 and $K3$. We have already seen in Section 6.2.1 that the elliptic genus of T^4 should vanish. We therefore omit that case here. We shall also categorically ignore Calabi–Yau manifolds that are tori for this reason. The elliptic genus for a $K3$ surface is the Jacobi form that appears in Mathieu moonshine. Its expansion in terms of $\mathcal{N}=4$ characters [**EOTY89**] is[4]

$$\mathbf{EG}_{K3}(\tau, z) = 2\phi_{0,1}(\tau, z) = 20\,\mathbf{ch}^{\mathcal{N}=4}_{2,0,0}(\tau, z) - 2\,\mathbf{ch}^{\mathcal{N}=4}_{2,0,\frac{1}{2}}(\tau, z) + \sum_{n=1}^{\infty} A_n \mathbf{ch}^{\mathcal{N}=4}_{2,n,\frac{1}{2}}(\tau, z)\,, \tag{6.6}$$

where all coefficients A_n are related to the dimensions of irreducible representations of $\mathbf{M}_{24}$. The above expansion of the elliptic genus is in terms of $\mathcal{N}=4$ characters following the original work [**EOT11**]. This can be done since $K3$ is a hyper-Kähler manifold and the superconformal worldsheet theory has $\mathcal{N}=(4,4)$ supersymmetry. However, we can also expand the elliptic genus of $K3$ in terms of the $\mathcal{N}=2$ characters given in [**EH10a**] [5]

$$\mathbf{EG}_{K3}(\tau, z) = 2\phi_{0,1}(\tau, z) = -20\,\mathbf{ch}^{\mathcal{N}=2}_{2,0,0}(\tau, z) + 2\,\mathbf{ch}^{\mathcal{N}=2}_{2,0,1}(\tau, z) - \sum_{n=1}^{\infty} A_n \mathbf{ch}^{\mathcal{N}=2}_{2,n,1}(\tau, z)\,, \tag{6.7}$$

where A_n are still the coefficients of the irreducible representations of $\mathbf{M}_{24}$. The overall negative sign is a choice of convention as in [**EH10a**].

There is also a moonshine phenomenon that connects Enriques surfaces (an involution of the $K3$ surface) to $\mathbf{M}_{12}$ [**EH13**]. The elliptic genus for an Enriques surface

[4]See Appendix A for the explicit formulas for the $\mathcal{N}=2$ and $\mathcal{N}=4$ characters. The subscripts on the characters $\mathbf{ch}_{d,h-c/24,\ell}$ are the complex dimension d of the Calabi–Yau that determines the central charge to be $c=3d$, the eigenvalue $h-c/24$ of $L_0-c/24$ and the eigenvalue ℓ of J_0.

[5]For the particular case of $c=6$, i.e. for $d=2$, the $\mathcal{N}=2$ superconformal algebra extended by spectral flow generators is the same as the $\mathcal{N}=4$ superconformal algebra. This means that in this case the $\mathcal{N}=2$ and the $\mathcal{N}=4$ algebras are the same (up to an overall sign in our conventions).

is $\mathbf{EG}_{\text{Enr}}(\tau,z) = \frac{1}{2}\mathbf{EG}_{K3}(\tau,z)$, since $\chi_{\text{Enr}} = 12 = \mathbf{EG}_{\text{Enr}}(\tau,0) = \phi_{0,1}(\tau,0)$. This leads to the following expansion in terms of $\mathcal{N}=4$ characters

$$\mathbf{EG}_{\text{Enr}}(\tau,z) = \phi_{0,1}(\tau,z) = 10\,\mathbf{ch}^{\mathcal{N}=4}_{2,0,0}(\tau,z) - 1\,\mathbf{ch}^{\mathcal{N}=4}_{2,0,\frac{1}{2}}(\tau,z) + \sum_{n=1}^{\infty}\frac{A_n}{2}\mathbf{ch}^{\mathcal{N}=4}_{2,n,\frac{1}{2}}(\tau,z)\,. \tag{6.8}$$

Note that all the A_n are even so that also in this case the expansion coefficients are always integers and one can again decompose the coefficients into irreducible representations of $\mathbf{M}_{12}$:

$$\begin{aligned} 10 &= 11-1\,,\\ -1 &= -1\,,\\ A_1 &= 45\,,\\ A_2 &= 55+176\,,\\ A_3 &= 66+2\cdot 120+2\cdot 144+176\,,\\ &\dots\,. \end{aligned} \tag{6.9}$$

We can of course again expand the elliptic genus $\mathbf{EG}_{\text{Enr}}(\tau,z)$ in $\mathcal{N}=2$ characters, which leads to the same expansion coefficients but with an overall minus sign.

6.2.3. Calabi–Yau 3–folds. Expanding the elliptic genus of Calabi–Yau 3–folds in terms of $\mathcal{N}=2$ characters does not prove a fruitful exercise in moonshine. The elliptic genus is given as

$$\mathbf{EG}_{CY_3}(\tau,z) = \frac{\chi_{CY_3}}{2}\,\phi_{0,\frac{3}{2}} = \frac{\chi_{CY_3}}{2}\left(\mathbf{ch}^{\mathcal{N}=2}_{3,0,\frac{1}{2}}(\tau,z) + \mathbf{ch}^{\mathcal{N}=2}_{3,0,-\frac{1}{2}}(\tau,z)\right)\,, \tag{6.10}$$

where the overall normalization is fixed in terms of (6.3) and (3.47). Most expansion coefficients are zero, for the case of the CY_3. We wish to point out that while we study CY 3–folds only very briefly in this thesis, the simplicity of the elliptic genus does not mean that they are of reduced interest. Contrary to this, the study of enumerative invariants of CY_3 is a challenging problem in mathematical physics [**KS08**, **DM11**, **CK00**]. For example, it was shown in [**CDD+13**] that the Gromov–Witten invariants of certain CY_3 manifolds are connected to Mathieu moonshine. This was further explored and studied in [**Wra14**, **PW14**, **DDL16**, **CD17b**, **CD17a**, **BCKS20**] and explicit CY_3 models obeying such relations were constructed in [**BCKS20**].

6.2.4. Calabi–Yau 4–folds. For Calabi–Yau 4–folds the corresponding space $J_{0,2}$ of Jacobi forms is generated by two basis elements and one can fix the coefficients in terms of the Euler number χ_{CY_4} and $\chi_0 = \sum_r (-1)^r h^{0,r}$ using (3.47). One then finds

$$\mathbf{EG}_{CY_4}(\tau,z) = \frac{\chi_{CY_4}}{144}\left(\phi_{0,1}^2 - E_4\,\phi_{-2,1}^2\right) + \chi_0 E_4\,\phi_{-2,1}^2\,. \tag{6.11}$$

$\chi_0 = h^{0,0} + h^{0,4} = 2$ for genuine CY 4–folds.[6]

There are a variety of interesting connections to sporadic groups that have already appeared in literature. The first and somewhat trivial case is the product of two $K3$ manifolds whose elliptic genus is given by $\mathbf{EG}_{K3\times K3}(\tau,z) = 4\phi_{0,1}^2$. This function exhibits an $\mathbf{M}_{24}\times\mathbf{M}_{24}$ symmetry and one could ask if whether such a symmetry (or an $\mathbf{M}_{12}\times\,\mathbf{M}_{24}$ symmetry for $2\phi_{0,1}^2$) is realized by certain genuine Calabi–Yau 4–folds. The Jacobi form $\frac{1}{24}\left(\phi_{0,1}^2 - E_4\,\phi_{-2,1}^2\right)$ appears in the context of Umbral moonshine [**CDA14**]

[6] A genuine Calabi–Yau d–manifold is one for which the holonomy group is $SU(d)$. A non-genuine Calabi–Yau d–manifold is one for which the holonomy group is a subgroup of $SU(d)$.

and exhibits a $2.\mathbf{M}_{12}$ moonshine when expanded in $\mathcal{N}=4$ characters, In [EH12], an $L_2(11)$ moonshine was established upon expanding this function in terms of $\mathcal{N}=2$ characters. In [CDD$^+$14], it was shown that the function $\frac{1}{6}(\phi_{0,1}^2+5E_4\,\phi_{-2,1}^2)$ exhibits $\mathbf{M}_{22}$ moonshine when expanded in $\mathcal{N}=4$ characters and $\mathbf{M}_{23}$ moonshine when expanded in $\mathcal{N}=2$ characters.[7] All of these and further potential connections to sporadic groups will be discussed in future work [Kid] where twinings will also be studied in great detail.

6.2.5. Calabi–Yau 5–folds. As discussed in Section 6.1 and (6.2), the elliptic genus for a Calabi–Yau manifold with odd complex dimensions d can be expressed in terms of the same functions as those occurring in the expression for the elliptic genus of a Calabi–Yau manifold with complex dimension $d-3$, since Jacobi forms of half-integral index can be written as the product of $\phi_{0,\frac{3}{2}}$ times an integral index Jacobi form $J_{0,\frac{d}{2}}=\phi_{0,\frac{3}{2}}J_{0,\frac{d-3}{2}}$ [Gri99]. This means that the elliptic genus for CY 5–folds is proportional to $\phi_{0,\frac{3}{2}}\phi_{0,1}$ and we can fix the prefactor in terms of the Euler characteristic χ_{CY_5}. We also recall that the $\mathcal{N}=4$ characters for central charge $c=3d$, multiplied by $\phi_{0,\frac{3}{2}}$, can be expressed in terms of $\mathcal{N}=2$ characters for central charge $c=3(d+3)$ (see appendix A). We find the following relations

$$
\begin{aligned}
\phi_{0,\frac{3}{2}}\mathbf{ch}^{\mathcal{N}=4}_{2,0,0} &= -\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}-\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}\,,\\
\phi_{0,\frac{3}{2}}\mathbf{ch}^{\mathcal{N}=4}_{2,0,\frac{1}{2}} &= -\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}-\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}+\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}\,,\\
\phi_{0,\frac{3}{2}}\mathbf{ch}^{\mathcal{N}=4}_{2,n,\frac{1}{2}} &= -\mathbf{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}-\mathbf{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}\,,\qquad \forall n=1,2,\dots\,.
\end{aligned}
\tag{6.12}
$$

This means that CY 5–folds have the following expansion of the elliptic genus in terms of $\mathcal{N}=2$ characters:

$$
\begin{aligned}
\mathbf{EG}_{CY_5}(\tau,z) &= \frac{\chi_{CY_5}}{24}\phi_{0,\frac{3}{2}}\phi_{0,1}\\
&= -\frac{\chi_{CY_5}}{48}\Bigg[22\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)-2\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\
&\qquad +\sum_{n=1}^{\infty}A_n\left(\mathbf{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right)\Bigg]\,.
\end{aligned}
\tag{6.13}
$$

For CY 5–folds with $\chi_{CY_5}=-48$ we find the same expansion coefficients as in Mathieu moonshine, while for $\chi_{CY_5}=-24$ we find the same coefficients as for Enriques moonshine. Since the overall sign in the definition of the $\mathcal{N}=2$ characters is a choice, we can conclude the same for CY 5–folds with $\chi_{CY_5}=48$ and $\chi_{CY_5}=24$. The decomposition of all the coefficients into the dimensions of irreducible representations is identical to the ones given above in equations (5.16) and (6.9) except for the first coefficient for which we have $22=23-1$ for $\mathbf{M}_{24}$ and $11=11$ for $\mathbf{M}_{12}$. This follows from the relations between $\mathcal{N}=4$ and $\mathcal{N}=2$ characters given above in (6.12).

A trivial class of examples where the elliptic genus will exhibit Mathieu or Enriques moonshine is given by manifolds that are products of $K3$ or Enriques surfaces with

[7] This function also exhibits $\mathbf{M}_{24}$ moonshine when expanded in extended $\mathcal{N}=1$ characters [CHKW15]. There are also other groups that can arise instead of the Mathieu groups but the later are somewhat special [CDD$^+$14, CHKW15]. When evaluated at torsion points $y=-1$ and $y=\pm\sqrt{q}$, the same function gives rise to Conway moonshine [FLM85, Dun05], the moonshine phenomenon associated with the Conway group.

CY 3–folds. While these cases are rather trivial, we see from the above that any CY 5-fold could in principle exhibit a connection to a Mathieu group. This is therefore the reason we choose to analyze CY 5–folds.

6.2.6. Calabi–Yau 6–folds. The elliptic genus for Calabi–Yau 6–folds is determined uniquely in terms of three topological numbers which are the Euler number χ_{CY_6} and $\chi_p = \sum_{r=0}^{6}(-1)^r h^{p,r}$ for $p = 0, 1$. The elliptic genus for CY_6 is given by the expression

$$\begin{aligned}\mathbf{EG}_{CY_6}(\tau,z) = \frac{\chi_{CY_6}}{1728}\phi_{0,1}^3 - \frac{1}{576}(\chi_{CY_6} - 48(\chi_1 + 6\chi_0))E_4\,\phi_{-2,1}^2\,\phi_{0,1} \\ -\frac{1}{864}(\chi_{CY_6} - 72(\chi_1 - 6\chi_0))E_6\,\phi_{-2,1}^3\,. \end{aligned} \tag{6.14}$$

One can again find many interesting but simple cases by taking triple products of $K3$ or Enriques surfaces or by considering $K3 \times CY_4$. These cases trivially exhibit a variety of potential connections to sporadic groups like $\mathrm{L}_2(11)$, $\mathbf{M}_{12}$, $\mathbf{M}_{22}$, $\mathbf{M}_{23}$ and $\mathbf{M}_{24}$ (c.f. Section 6.2.4). A potential connection between CY 6–folds and sporadic groups therefore seems plausible, but is yet to be understood.

For a special case of CY_6, when $\chi_0 = \chi_1 = 0$, we find that

$$\begin{aligned}\mathbf{EG}_{CY_6}(\tau,z) &= \frac{\chi_{CY_6}}{1728}\left(\phi_{0,1}^3 - 3E_4\,\phi_{-2,1}^2\,\phi_{0,1} - 2E_6\,\phi_{-2,1}^3\right) \\ &= \frac{\chi_{CY_6}}{4}\phi_{0,\frac{3}{2}}^2 \\ &= \chi_{CY_6}\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \\ &= \frac{\chi_{CY_6}}{8}\Big[4\mathbf{ch}_{6,0,0}^{N=4} + \Big(-2\mathbf{ch}_{6,0,\frac{1}{2}}^{N=4} + 14\mathbf{ch}_{6,1,\frac{1}{2}}^{N=4} + 42\mathbf{ch}_{6,2,\frac{1}{2}}^{N=4} + 86\mathbf{ch}_{6,3,\frac{1}{2}}^{N=4} + \dots\Big) \\ &\qquad - \left(16\mathbf{ch}_{6,1,1}^{N=4} + 48\mathbf{ch}_{6,2,1}^{N=4} + 112\mathbf{ch}_{6,3,1}^{N=4} + \dots\right) \\ &\qquad + \Big(6\mathbf{ch}_{6,1,\frac{3}{2}}^{N=4} + 28\mathbf{ch}_{6,2,\frac{3}{2}}^{N=4} + 56\mathbf{ch}_{6,3,\frac{3}{2}}^{N=4} + \dots\Big)\Big]\,. \end{aligned} \tag{6.15}$$

For $\chi_{CY_6} = 8$ and the corresponding expansion in terms of $\mathcal{N} = 4$ characters, the above Jacobi form is related to $2.AGL_3(2)$ [**CDA14**] via the Umbral moonshine conjecture which was later proven in [**DGO15**]. This is similar to the case of CY 4–folds that seem to be related to Umbral moonshine for $\chi_0(CY_4) = 0$. Here, however, we can easily find spaces that have the above elliptic genus (6.15). Since the product of two CY 3–folds gives a complex six dimensional manifold with $\chi_0 = \chi_1 = 0$, the resulting elliptic genus is as above i.e., $\chi_{CY_6} = \chi_{CY_3^{(1)}} \cdot \chi_{CY_3^{(2)}}$. For Umbral moonshine, the requirement is an elliptic genus given in (6.15) for the case of $\chi_{CY_6} = 8$. Therefore, the product of a CY 3-fold with Euler number ± 2 with another CY 3-fold with Euler number ± 4 is an excellent candidate to study Umbral moonshine in CY 6–folds.

6.2.7. Comments on Calabi–Yau manifolds of dimension $d > 6$. In the study of higher dimensional CY manifolds, there exists the possibility of making connections to Umbral moonshine [**CDA14**]. Since we focus on the case of Mathieu moonshine in the description of this thesis, we shall not go further into Umbral moonshine. We however present a short comment.

Due to the relationship between elliptic genera of lower dimensional CY manifolds and higher dimensional ones, and the increasing number of CY manifolds with dimension, it therefore is clear that the possibility of having connections to sporadic groups grows as we increase the complex dimension of the CY manifold. There is however no simple way of studying these cases since a systematic construction of CY$_{d>6}$ manifolds is lacking, in the spirit and sense of there being a construction for lower dimensional CY manifold. Nevertheless we can check under what conditions the extremal Jacobi forms that appear in Umbral moonshine [**CDA14, CDH13**] can arise as elliptic genera of Calabi–Yau manifolds or products thereof.[8]

Recall that the elliptic genus of $K3$, i.e., a Jacobi form of weight 0 and index 1, provides the first example of Umbral moonshine. As we mentioned above, the index 2 Jacobi form that arises in Umbral moonshine would correspond to the elliptic genus of a CY 4–fold with $\chi_0(CY_4) = 0$. However, any genuine CY d-fold satisfies

$$\chi_0(CY_d) = \begin{cases} 0 & \text{if } d \text{ is odd,} \\ 2 & \text{if } d \text{ is even.} \end{cases} \tag{6.16}$$

We cannot therefore recover the Jacobi form of weight 0 and index 2 form from a genuine CY 4–fold, nor from $K3 \times K3$ since $\chi_0(K3 \times K3) = \chi_0(K3)^2 = 4$. Likewise, $T^4 \times K3$ has vanishing χ_0 since $\chi_0(T^4) \cdot \chi_0(K3) = 0 \cdot 2 = 0$ but also vanishing elliptic genus. Above we have seen, however, that six complex dimensional Calabi–Yau spaces with $\chi_0 = \chi_1 = 0$ have as elliptic genus the Jacobi form of weight 0 and index 3 that appears in Umbral moonshine. In particular the product of any two CY 3–folds has $\chi_0 = \chi_1 = 0$.[9] The other interesting Jacobi forms in [**CDA14**] have weight zero and index 4, 6 and 12. We first look at the case of $d = 8$ that leads to a Jacobi form with index 4. The relevant form in Umbral moonshine arises for $\chi_0 = \chi_1 = \chi_2 = 0$. We can again ask whether we can get this from a product of CY manifolds. This is however not the case. For the case of CY$_5\times$CY$_3$ we find $\chi_0 = \chi_1 = 0$ but $\chi_2(CY_3 \times CY_5) \propto \chi(CY_3) \cdot \chi(CY_5)$. So only if either $\chi(CY_3) = 0$ or $\chi(CY_5) = 0$, do we satisfy the required property. However, in this case we have $\mathbf{EG}_{CY_3 \times CY_5}(\tau, z) \propto \chi(CY_3) \cdot \chi(CY_5) = 0$. Similarly for the case $K3 \times CY_3 \times CY_3'$ we find $\chi_0 = \chi_1 = 0$ but $\chi_2(K3 \times CY_3 \times CY_3') \propto \chi(CY_3) \cdot \chi(CY_3')$ and $\mathbf{EG}_{K3 \times CY_3 \times CY_3'}(\tau, z) \propto \chi(CY_3) \cdot \chi(CY_3')$ and there is again no non-trivial solution.

The same holds true for $d = 12$ in which case the Umbral moonshine Jacobi form arises as elliptic genus for spaces with $\chi_0 = \chi_1 = \chi_2 = \chi_3 = \chi_4 = \chi_6 = 0$ which is very restrictive and cannot be realized by taking products of Calabi–Yau manifolds. The most promising case, which is the product of four CY 3–folds, has $\chi_0 = \chi_1 = \chi_2 = \chi_3 = 0$ but $\chi_4 \neq 0 \neq \chi_6$ unless the Euler number of one of CY 3–folds vanishes in which case the entire elliptic genus is zero as well.

The original Umbral moonshine was extended in [**CDH13**] and there are many further Jacobi forms that could in principle arise from products of CY manifolds. However, again this does not seem possible for the cases analysed here. For example, there is an extremal Jacobi form that could arise from the elliptic genus of a CY 10-fold with $\chi_0 = \chi_1 = \chi_2 = \chi_3 = 0$, but again this cannot be realized by taking products of, for example, two $K3$ surfaces and two CY 3–folds. Similarly the index 9 extremal Jacobi form in Umbral moonshine would correspond to a Calabi–Yau manifold with $\chi_p = 0,\ \forall p \leq 7$, but the product of six CY 3–folds has $\chi_p = 0,\ \forall p \leq 5$, and $\chi_6 \neq 0$, if we require the elliptic genus of this product to be non-vanishing. So it seems that only

[8]Extremal Jacobi forms are those Jacobi forms that can be decomposed into $\mathcal{N} = 4$ superconformal characters [**CDA14**].

[9]Recall the useful formula $\chi_i(CY \times CY') = \sum_{j=0}^{i} \chi_j(CY) \cdot \chi_{i-j}(CY')$.

the extremal Jacobi forms of index 1 and 3 can arise as elliptic genera of Calabi–Yau manifolds or their products.

6.3. Twined elliptic genera from localization

We now turn to the key idea of the calculation. We have seen before that the analogues of the McKay–Thompson series for Mathieu moonshine are related to the (twined)–elliptic genera of the Calabi–Yau manifold. In this section, we present an overview of how the elliptic genera (with twining) can be constructed for generic CY 5–folds. We can speculate by analogy with $K3$ we expect not all Calabi–Yau 5–folds with $\chi_{CY_5} = -48$ to have an $\mathbf{M}_{24}$symmetry at every, or any, point in their moduli space. However, the elliptic genus could see some combined symmetry group that arises, for example, from different points in moduli space via symmetry surfing, again in analogy with $K3$ [**TW15, TW13a, GKP17**]. On the other hand it is not impossible that some CY 5–folds have a genuine $\mathbf{M}_{24}$ symmetry at some point in the moduli space and that this could then explain the corresponding expansion of the elliptic genus. It would then provide a higher dimensional origin for Mathieu moonshine. Something similar could happen for the other cases that involve Calabi–Yau 4–folds and 6–folds but we shall not discuss them here. We proceed by explicitly calculating twined elliptic genera for many different Calabi–Yau manifolds that are hypersurfaces in weighted projective spaces. The construction of CY manifolds that are hypersurfaces in weighted projective space can be studied in Chapter 14.5 in [**BLT13**] and in [**KS92**].

6.3.1. Calculating the twined elliptic genus. To calculate the twined elliptic genus, we use the localization techniques developed in [**BEHT14**] (first applied to moonshine in [**HKP14**]) to calculate elliptic genera twined by a symmetry of the Calabi–Yau manifold. The Calabi–Yau manifolds we are interested in are hypersurfaces in weighted projective ambient spaces. A particular Calabi–Yau d-fold that is a hypersurface in the weighted projective space $\mathbb{CP}^{d+1}_{w_1\ldots w_{d+2}}$ is determined by a solution of $p(\Phi_1,\ldots,\Phi_{d+2}) = 0$, where the Φ_i denote the homogeneous coordinates of the weighted projective space and p is a transverse polynomial of degree $m = \sum_i w_i$.

Following the techniques in [**BEHT14**], we consider a two-dimensional gauged linear sigma model with $\mathcal{N} = (2,2)$ supersymmetry. We have a $U(1)$ gauge field under which the chiral multiplets Φ_i have charge w_i. Additionally we have one extra chiral multiplet X with $U(1)$ charge $-m$. The superpotential that is invariant under $U(1)$ gauge transformations is given by $W = Xp(\Phi_1,\ldots,\Phi_{d+2})$. Then the F-term equation $\partial W/\partial X = p = 0$ restricts us to the Calabi–Yau hypersurface above. We can assign $\mathcal{R}$-charge zero to the Φ_i and 2 to the chiral multiplet X which ensures that the superpotential has always the correct $\mathcal{R}$-charge 2.

One can define a refined elliptic genus, depending on the extra chemical potential $x = e^{2\pi i u}$, as

$$\mathbf{EG}_{\text{ref}}(\tau,z,u) = \mathrm{Tr}_{RR}\left((-1)^{F_L} y^{J_0} q^{L_0-\frac{d}{8}} x^Q (-1)^{F_R} \bar{q}^{\bar{L}_0-\frac{d}{8}}\right). \tag{6.17}$$

This refined elliptic genus also keeps track of the $U(1)$ charges Q of the states in the theory. Each chiral multiplet of $U(1)$ charge Q and $\mathcal{R}$-charge R gives a multiplicative contribution to this refined elliptic genus that is

$$\mathbf{EG}^{\Phi}_{\text{ref}}(\tau,z,u) = \frac{\vartheta_1\left(\tau,\left(\frac{R}{2}-1\right)z+Qu\right)}{\vartheta_1\left(\tau,\frac{R}{2}z+Qu\right)}. \tag{6.18}$$

The Abelian vector field gives rise to a (u independent) factor

$$\mathbf{EG}^{vec}_{\mathrm{ref}}(\tau,z) = \frac{i\eta(\tau)^3}{\vartheta_1(\tau,-z)}\,. \tag{6.19}$$

Combining these we find for for the case of our interests with

(1) $d+2$ chiral multiplets Φ_i with $U(1)$ charge w_i and zero $\mathcal{R}$-charge,
(2) one chiral multiplet X with $U(1)$ charge $-m$ and $\mathcal{R}$-charge 2
(3) and one Abelian vector multiplet,

we obtain the following refined elliptic genus

$$\mathbf{EG}_{\mathrm{ref}}(\tau,z,u) = \frac{i\eta(\tau)^3}{\vartheta_1(\tau,-z)}\frac{\vartheta_1(\tau,-mu)}{\vartheta_1(\tau,z-mu)}\prod_{i=1}^{d+2}\frac{\vartheta_1(\tau,-z+w_iu)}{\vartheta_1(\tau,w_iu)}\,. \tag{6.20}$$

The standard elliptic genus is obtained by integrating over u. This integral localizes to a sum of contour integrals [BEHT14] so that we have

$$\mathbf{EG}_{CY_d}(\tau,z) = \frac{i\eta(\tau)^3}{\vartheta_1(\tau,-z)}\sum_{u_j\in\mathcal{M}^-_{sing}}\oint_{u=u_j} du\frac{\vartheta_1(\tau,-mu)}{\vartheta_1(\tau,z-mu)}\prod_{i=1}^{d+2}\frac{\vartheta_1(\tau,-z+w_iu)}{\vartheta_1(\tau,w_iu)}\,, \tag{6.21}$$

where $\mathcal{M}^-_{sing}$ is the space of poles of the integrand where the chiral multiplets become massless. For a chiral multiplet with $U(1)$ charge Q and $\mathcal{R}$-charge R these singularities are located at

$$Qu + \frac{R}{2}z = 0\,,\quad \mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z}\,. \tag{6.22}$$

In the above formula one can restrict to the singularities for chiral multiplets with $Q<0$, hence the superscript $\mathcal{M}^-_{sing}$. In our particular setup only the chiral multiplet X has negative $U(1)$ charge so the singularities are solutions to

$$-mu + z = -k - \ell\tau\,,\quad k,\ell\in\mathbb{Z}\,. \tag{6.23}$$

The integrand above is periodic under the identification $u\sim u+1\sim u+\tau$ and the solutions within one fundamental domain of u are

$$u = (z+k+\ell\tau)/m\,,\quad 0\le k,\ell<m\,. \tag{6.24}$$

We can rewrite the above expression as

$$\mathbf{EG}_{CY_d}(\tau,z) = \frac{i\eta(\tau)^3}{\vartheta_1(\tau,-z)}\sum_{k,\ell=0}^{m-1}\oint_{u=(k+\ell\tau+z)/m} du\frac{\vartheta_1(\tau,-mu)}{\vartheta_1(\tau,z-mu)}\prod_{i=1}^{d+2}\frac{\vartheta_1(\tau,-z+w_iu)}{\vartheta_1(\tau,w_iu)}\,. \tag{6.25}$$

The above can be further simplified by using properties of the ϑ-function [BEHT14] and this leads to the following simple formula for the elliptic genus of a Calabi–Yau d–fold that is a hypersurface in a weighted projective space and that can be described by a transverse polynomial:

$$\begin{aligned}\mathbf{EG}_{CY_d}(\tau,z) &= \sum_{k,\ell=0}^{m-1}\frac{e^{-2\pi i\ell z}}{m}\prod_{i=1}^{d+2}\frac{\vartheta_1\left(\tau,\frac{w_i}{m}(k+\ell\tau+z)-z\right)}{\vartheta_1\left(\tau,\frac{w_i}{m}(k+\ell\tau+z)\right)}\\ &= \sum_{k,\ell=0}^{m-1}\frac{y^{-\ell}}{m}\prod_{i=1}^{d+2}\frac{\vartheta_1\left(q,e^{\frac{2\pi i w_i k}{m}}q^{\frac{w_i\ell}{m}}y^{\frac{w_i}{m}-1}\right)}{\vartheta_1\left(q,e^{\frac{2\pi i w_i k}{m}}q^{\frac{w_i\ell}{m}}y^{\frac{w_i}{m}}\right)}\,.\end{aligned} \tag{6.26}$$

If we want to twine the elliptic genus by an Abelian symmetry that is generated by an element g acting via

$$g : \Phi_i \to e^{2\pi i \alpha_i} \Phi_i \,, \qquad i = 1, 2, \ldots, d+2 \,, \tag{6.27}$$

then this leads to a shift of the original z coordinate (i.e. the second argument) of the ϑ_1-functions for each Φ_i by α_i. The resulting twined elliptic genus is therefore given by

$$\begin{aligned} \mathbf{EG}^{(g)}_{CY_d}(\tau, z) &= \mathrm{Tr}_{RR} \left(g\, (-1)^{F_L} y^{J_0} q^{L_0 - \frac{d}{8}} (-1)^{F_R} \bar{q}^{\bar{L}_0 - \frac{d}{8}} \right) \\ &= \sum_{k,\ell=0}^{m-1} \frac{e^{-2\pi i \ell z}}{m} \prod_{i=1}^{d+2} \frac{\vartheta_1\left(\tau, \alpha_i + \frac{w_i}{m}(k + \ell\tau + z) - z\right)}{\vartheta_1\left(\tau, \alpha_i + \frac{w_i}{m}(k + \ell\tau + z)\right)} \\ &= \sum_{k,\ell=0}^{m-1} \frac{y^{-\ell}}{m} \prod_{i=1}^{d+2} \frac{\vartheta_1\left(q, e^{2\pi i \left(\alpha_i + \frac{w_i k}{m}\right)} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m} - 1}\right)}{\vartheta_1\left(q, e^{2\pi i \left(\alpha_i + \frac{w_i k}{m}\right)} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m}}\right)} \,. \end{aligned} \tag{6.28}$$

The method of [**BEHT14**] can also be used for non-Abelian permutation symmetries. In this case one can perform a coordinate transformation that diagonalizes the permutation matrix that acts on the Φ_i. In this new basis the action is then again Abelian and the phase factors are the eigenvalues of the permutation matrix.

6.4. Analysis for Calabi–Yau 5–folds

From the previous sections, we have seen that, up to an overall constant, all CY 5–folds allow for a character expansion that is essentially the same as the expansion of the elliptic genus of $K3$ in the discovery of Mathieu moonshine. We now understand the implications of Mathieu moonshine for the construction of CY 5–folds by using the techniques of symmetry surfing i.e., by explicitly computing the twined elliptic genus for many CY 5–folds and studying their $\mathcal{N} = 2$ character expansions.

6.4.1. Description of computational algorithm. The starting point is to obtain a database of CY 5–folds. This can be obtained from [**KS**]. In total, we start with a list of 5 757 727 CY 5–folds that can be described by reflexive polytopes. However, out of these 5 757 727 CY 5–folds only 19 353 are described by transverse polynomials in weighted projective spaces. We recall that the construction of CY_5 as hypersurfaces in weighted projective space is a requirement to compute the elliptic genus using localization techniques.

We may now calculate the twined elliptic genus as a power series in q for these examples to get a better understanding of a potential connection to a sporadic group. Based on the elliptic genus in equation (6.13)

$$\begin{aligned} \mathbf{EG}_{CY_5}(\tau, z) = -\frac{\chi_{CY_5}}{48} \Bigg[& 22 \left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau, z) + \mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau, z) \right) - 2 \left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau, z) + \mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau, z) \right) \\ & + \sum_{n=1}^{\infty} A_n \left(\mathbf{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau, z) + \mathbf{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau, z) \right) \Bigg] \,, \end{aligned} \tag{6.29}$$

we look for a genuine CY 5-fold whose elliptic genus transforms covariantly (up to a constant) like a function related to Mathieu moonshine. This happens in particular for the product manifold $\mathrm{CY}_3 \times K3$, where the elliptic genus is the product of a prefactor χ_{CY_3} and $\phi_{0,1}(\tau,z)$ which is a function that exhibits Mathieu moonshine.

For genuine CY 5–folds we find this behaviour. For the hypersurface in the weighted projective space $\mathbb{CP}^6_{1,1,1,3,5,9,10}$ we have

$$\begin{aligned}\mathbf{EG}_{CY_5}(\tau,z) = 3556\cdot \Bigg[&22\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)-2\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &+90\left(\mathbf{ch}^{\mathcal{N}=2}_{5,1,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,1,-\frac{3}{2}}(\tau,z)\right)+\dots\Bigg],\end{aligned} \tag{6.30}$$

corresponding to an Euler number of $\chi = -170\,688$. Under the $\mathbb{Z}_2$ symmetry

$$\mathbb{Z}_2 : \begin{cases} \Phi_1 \to -\Phi_1\,, \\ \Phi_2 \to -\Phi_2\,, \end{cases} \tag{6.31}$$

we find the twined elliptic genus

$$\begin{aligned}\mathbf{EG}^{tw,2\mathrm{A}}_{CY_5}(\tau,z) = 14\cdot \Bigg[&2\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)-2\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &+6\left(\mathbf{ch}^{\mathcal{N}=2}_{5,1,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,1,-\frac{3}{2}}(\tau,z)\right)+\dots\Bigg],\end{aligned} \tag{6.32}$$

which is a constant, 14, multiplied by the 2A series of $\mathbf{M}_{24}$. The constant contribution of 14 can be obtained by decomposing the constant appearing in front of the untwined elliptic genus (3556) in such a way that one gets 14 upon twining by the 2A element. It therefore seems plausible that there is an action of $\mathbf{M}_{24}$ acting on the elliptic genus of this particular CY 5-fold. Since the elliptic genus is an index, this symmetry could arise at certain points in moduli space or everywhere in moduli space or via symmetry surfing. This possibility extends to other CY 5–folds, where simple low order twinings agree with the expectation. Here we list a few interesting examples:

Applying the same order two twining as the one given in equation (6.31) to the Calabi–Yau hypersurface in the weighted projective space $\mathbb{CP}^6_{1,2,2,3,4,4,8}$ results again in the 2A element of $\mathbf{M}_{24}$, but now with a half integral prefactor of 69/2. This might seem concerning at first i.e., to have a non–integer prefactor but we could remedy this by restricting ourselves to $\mathbf{M}_{12}$ instead.

We can likewise study order four twinings by the $\mathbb{Z}_4$ symmetry that acts as

$$\mathbb{Z}_4 : \begin{cases} \Phi_1 \to i\,\Phi_1\,, \\ \Phi_2 \to -i\,\Phi_2\,. \end{cases} \tag{6.33}$$

For the Calabi–Yau hypersurface in $\mathbb{CP}^6_{1,1,1,1,4,4,4}$ under $\mathbb{Z}_4$ twining, we find the 4B series with prefactor 42 [**BCK+18**]. Similarly, one can find more specific examples where particular symmetries for particular CY 5–folds give the expected twining function of Mathieu moonshine up to an overall prefactor.

We now perform a more systematic study of twinings of small order for CY 5–folds that are hypersurfaces in weighted projective spaces for the list of CY 5–folds that can be described as such as in [**KS**]:

(1) We start with the 19 353 CY 5–folds that can be described by a transverse polynomial in the homogeneous coordinates of the ambient weighted projective space.

(2) We then solve for a single transverse polynomial computationally. Due to the difficulty in constructing the transverse polynomials, we find a transverse polynomial for only 18 880 CY 5–folds.
(3) For each of these 18 880 CY 5–folds, we check if whether there is an Abelian $\mathbb{Z}_2$ symmetry acting on them and found that 16 727 of the manifolds have at least one such symmetry.
(4) For these 16 727 manifolds, we calculate the twined elliptic genus for a single $\mathbb{Z}_2$ symmetry to zeroth order in q. Restrictions on the code to prevent processor overclocking led to analyzing a set of 13 642 twined elliptic genera. The computational complexity of the problem allows us to suspect that these CY 5–folds could likely be hypersurfaces in weighted projective spaces of low degree.

6.4.2. Results of analysis for Calabi–Yau 5–folds. We find that the order two twining by the $\mathbb{Z}_2$ symmetry always leads to a function that is a linear combination of the 1A and 2A series of $\mathbf{M}_{24}$ with prefactors that are integer or half integer. For example, when twining the the Calabi–Yau hypersurface in the weighted projective space $\mathbb{CP}^6_{1,1,1,1,1,1,3}$ by the symmetry in equation (6.31), we find the following twined elliptic genus

$$\begin{aligned}\mathbf{EG}^{tw}_{CY_5}(\tau,z) = \frac{9}{2}\cdot\Bigg[&22\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)-2\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &+90\left(\mathbf{ch}^{\mathcal{N}=2}_{5,1,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,1,-\frac{3}{2}}(\tau,z)\right)+\mathcal{O}(q^2)\Bigg]\\ +43\cdot\Bigg[&6\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)-2\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &-6\left(\mathbf{ch}^{\mathcal{N}=2}_{5,1,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,1,-\frac{3}{2}}(\tau,z)\right)+\mathcal{O}(q^2)\Bigg].\end{aligned}\tag{6.34}$$

It follows from standard CFT arguments [**Che10, GHV10b**] that any elliptic genus twined by a group element g has to be a Jacobi form $\phi^g_{0,m}$ with a potentially non-trivial multiplier under $\Gamma_0(|g|) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\middle|\, c=0 \mod |g|\right\}$, where $|g|$ denotes the order of g. This means that $\phi^g_{0,m}$ transforms like

$$\begin{aligned}\phi^g_{0,m}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) &= e^{\frac{2\pi icd}{|g|h_g}}e^{\frac{2\pi imcz^2}{c\tau+d}}\phi^g_{0,m}(\tau,z)\,, && \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(|g|)\,,\\ \phi^g_{0,m}(\tau,z+\lambda\tau+\mu) &= (-1)^{2m(\lambda+\mu)}e^{-2\pi im(\lambda^2\tau+2\lambda z)}\phi^g_{0,m}(\tau,z)\,, && \lambda,\mu\in\mathbb{Z}\,.\end{aligned}\tag{6.35}$$

Since $c/|g|$ and d are integers we find that the above transformation can only lead to a non-trivial multiplier, i.e., a non-trivial phase in the first transformation law, if $h_g \neq 1$. A list of the h_g for Mathieu moonshine, which is the relevant case for us, is given for example in table 2 in [**Gan16**]. In particular we see that for $1A$ and $2A$ we have $h_{1A} = h_{2A} = 1$, while for 2B one has $h_{2B} = 2$. This means that the 2B twined elliptic genus has a non-trivial multiplier.

For low order twinings, the twined elliptic genera of CY 5–folds are generated by very few basis elements. This follows from the fact that this is the case for Mathieu moonshine and $K3$ manifolds, as discussed for example in [CHVZ16]. The functions that appear in the study of CY 5–folds are simply the product of $\phi_{0,\frac{3}{2}}$ and the functions that appear for $K3$ [Gri99]. We are particularly interested in order two twinings and would like to know whether they match the expectation of Mathieu moonshine, i.e., whether the expansion coefficients agree with the 2A or 2B expectation. Generically, twined elliptic genera of order two transform under $\Gamma_0(2)$ often with a non-trivial multiplier. If there is no non-trivial multiplier, then the twined elliptic genus is a linear combination of the 1A element and the 2A element since these are the generators of all modular functions of $\Gamma_0(2)$ without multiplier and with weight zero and index $m = \frac{5}{2}$. If there is a non-trivial multiplier then we expect the answer to be proportional to the 2B element. However, this is never the case from our analysis [BCK+18]. The 2B case at the lowest order in the character expansion is in fact an artifact of the twined elliptic genus being a combination of the 1A and 2A elements. This was expected since we only consider geometric symmetries that cannot lead to a non-trivial multiplier, since the a non-trivial multiplier arises from the failure of level matching in the g-twisted sector. This failure to level match can only occur for symmetries that act asymmetrically on left- and right-movers i.e., non-geometric symmetries, which is not the case for our geometric symmetries. This means all the twined elliptic genera computed are linear combinations of the 1A and 2A elements of $\mathbf{M}_{24}$. Since each coefficient in the expansion of the elliptic genus counts states with a given mass and charge, all the expansion coefficients of the basis functions have to have integer coefficients. This means that for our geometric order two twinings, we expect the answer to be a linear combination with integer coefficients of the following basis functions

$$
\begin{aligned}
f_{1a}(\tau,z) = {} & 11\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z) + \mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right) - \left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z) + \mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right) \\
& +45\left(\mathbf{ch}^{\mathcal{N}=2}_{5,1,\frac{3}{2}}(\tau,z) + \mathbf{ch}^{\mathcal{N}=2}_{5,1,-\frac{3}{2}}(\tau,z)\right) + \dots\,, \qquad (6.36)
\end{aligned}
$$

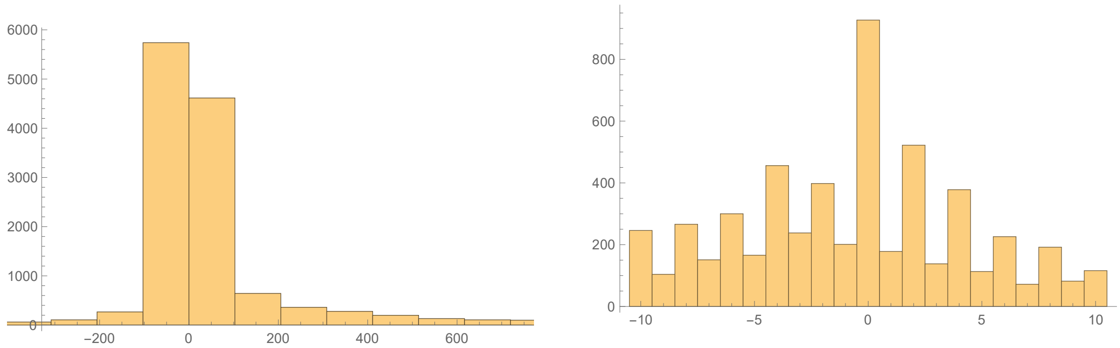
$$
\begin{aligned}
f_{2a}(\tau,z) = {} & 3\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z) + \mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right) - \left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z) + \mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right) \\
& -3\left(\mathbf{ch}^{\mathcal{N}=2}_{5,1,\frac{3}{2}}(\tau,z) + \mathbf{ch}^{\mathcal{N}=2}_{5,1,-\frac{3}{2}}(\tau,z)\right) + \dots\,. \qquad (6.37)
\end{aligned}
$$

These two basis function are $1/2$ times the expected 1A and 2A series from Mathieu moonshine, respectively and this arises because in Mathieu moonshine all coefficients in the character expansion are even integers.

Out of the 13 642 twined elliptic genera we find in 927 cases where the twined elliptic genus is proportional to f_{2a}. For 811 of these, the overall coefficient is an even integer and the twined elliptic genera are consistent with a possible $\mathbf{M}_{24}$ symmetry. Figure 7 shows two histograms of the coefficients of the f_{1a} function for the 13 642 twined elliptic genera. Most cases are of when the coefficients of f_{1a} are zero. Since a zero coefficient for f_{1a} is about twice as likely as a non–zero even integer coefficient, it still remains unclear if whether there is an action of $\mathbf{M}_{24}$.

We therefore analyze these 927 cases further:

(1) We generate a large number of $\mathbb{Z}_2$ symmetries for each of these 927 examples and calculate their elliptic genera for these different $\mathbb{Z}_2$.
(2) For almost all cases with multiple $\mathbb{Z}_2$ symmetries, there is only a single instance of an elliptic genus that is proportional to 2A. Several other order two twined elliptic genera are linear combinations of f_{1a} and f_{2a}. This excludes an action



(A) Histogram of all coefficients

(B) Histogram that is zoomed in on the zero peak.

FIGURE 7. A histogram of the coefficients of f_{1a} for the 13 642 twined elliptic genera. The coefficients peak near zero and as can be seen on the zoomed in histogram on the right this peaking is potentially larger than expected. We also see that even integer coefficients appear substantially more frequent.

of only $\mathbf{M}_{24}$ and supports the explanation that interesting twining genera for most of these 927 cases is indeed a fluke.

(3) For cases with only a single $\mathbb{Z}_2$ symmetry, we twine by a higher order element. For all cases we find at least one symmetry that leads to a twined elliptic genus that is not consistent with the $\mathbf{M}_{24}$ (or $\mathbf{M}_{12}$) expectation.

By twining the elliptic genus of CY 5–folds by order two and higher symmetries, we may exclude the possibility that any of the 13 642 CY 5–folds admit a strict $\mathbf{M}_{24}$ symmetry. These results are consistent with the expectation that CY 5–folds have symmetries that correspond to small discrete symmetry groups at certain points in moduli space but no large sporadic symmetry. However, we cannot exclude the more exotic possibility of symmetry groups that are even larger than $\mathbf{M}_{24}$.

Another possibility is that generic CY 5–folds have multiple copies of $\mathbf{M}_{24}$ as symmetry groups of their worldsheet sigma model superconformal field theories. So instead of interpreting the prefactor $-\frac{\chi_{CY_5}}{48}$ as a particular sum of irreducible representations of a single $\mathbf{M}_{24}$, we could have potentially up to $\left|\frac{\chi_{CY_5}}{48}\right|$ *distinct* $\mathbf{M}_{24}$ symmetries.[10] A geometric $\mathbb{Z}_2$ symmetry could then correspond to the 1A element in some of the $\mathbf{M}_{24}$'s and to the 2A element in other $\mathbf{M}_{24}$ groups and the prefactor could be a sum of non-trivial irreducible representations for some of the $\mathbf{M}_{24}$ symmetries. This would be consistent with all the results above (except for cases with half integral coefficients that would require us to replace $\mathbf{M}_{24}$ by $\mathbf{M}_{12}$). However, we do not entertain this possibility here due to its unlikelihood. The only cases where a check is rather straightforward are provided by the most interesting examples of CY 5–folds, namely the ones with Euler numbers $\chi(CY_5) = \pm 24$ and $\chi(CY_5) = \pm 48$. Based on the expansion of their elliptic genus, they could have a single $\mathbf{M}_{24}$ symmetry group for $\chi(CY_5) = \pm 48$ or a single $\mathbf{M}_{12}$ symmetry group for $\chi(CY_5) = \pm 24$, but these CY 5–folds become rare beyond a weight system of weight $m \sim 200$. The search for more such CY 5–folds using PALP

[10] Or analogously up to $\left|\frac{\chi_{CY_5}}{24}\right|$ distinct $\mathbf{M}_{12}$ symmetries.

Euler number	number of examples	cases with $\mathbb{Z}_2$ symmetries	reflexive cases
$\chi = -48$	72 (67)	64 (59)	6 (6)
$\chi = +48$	68 (59)	51 (43)	4 (4)
$\chi = -24$	32 (29)	26 (23)	4 (4)
$\chi = +24$	27 (24)	25 (22)	4 (4)

TABLE 2. The number of Calabi–Yau 5–folds constructed as hypersurfaces in weighted projective spaces. The values in parenthesis give the number of Calabi–Yau manifolds with different Hodge numbers.

[BKS+12] to go to larger weights m does not add much more to the list of relevant CY 5–folds.

6.4.3. Generating relevant CY 5–folds. In order to generate a few dozens of examples of CY 5–folds with Euler number ± 24 and ± 48 we first partition an integer m into seven integer weights w_i. We then check whether the Poincaré polynomial

$$P(x) = \prod_{i=1}^{7} \frac{1 - x^{m-w_i}}{1 - x^{w_i}} \tag{6.38}$$

evaluated at $x = 1$ is an integer and we consider only the ones for which this is true. Next, we apply the formula

$$\chi = \frac{1}{m} \sum_{k=1}^{m} \sum_{l=1}^{m} \prod_{gcd(l,k)\cdot \frac{w_i}{m} \in \mathbb{Z}} \frac{w_i - m}{w_i} \tag{6.39}$$

for the Euler number (see [KS92] for a derivation based on [Vaf89]) to the remaining cases and proceed only if the Euler numbers equal ± 24 or ± 48. Then the weights w_i may or may not correspond to a weighted projective space that admits a CY_5 hypersurface that is described by a transverse polynomial. For $7 \leq m \leq 600$, we generate all such weight systems and then determine explicitly using PALP [BKS+12] if whether these are indeed CY hypersurfaces. In this way, we can generate dozens of examples with small Euler numbers. In order to check whether these examples correspond to distinct manifolds we calculate all their Hodge numbers. The computation of the Hodge numbers demonstrates that there do exist manifolds with the same Hodge data but are described by hypersurfaces in different weighted projective spaces. For all the constructed CY 5–folds, we calculate all possible geometric $\mathbb{Z}_2$ symmetries using PALP. In Table 2 we list the number of manifolds using the technique described above, the number of examples with a $\mathbb{Z}_2$ symmetry and the number of those manifolds that can be described by reflexive polytopes.

We twine all of the above examples listen in Table 2 by their geometric $\mathbb{Z}_2$ symmetries.

6.4.4. Analysis on twined elliptic genera for constructed CY 5–folds. For the cases of $\chi = +48$ and $\chi = -48$ we find twined elliptic genera that are proportional to the 2A of $\mathbf{M}_{24}$ in 3.4% and 5.2% of the cases, respectively. We find that the coefficients f_{1A} demonstrate no particular peaking c.f. Figure 7b. Furthermore, there is no preference for even coefficients. It is worth noting that the cases for which the twined elliptic genus is proportional to the 2A expectation have prefactors with absolute values larger than one. Recall that these manifolds have an elliptic genus whose form is given by

$$\begin{aligned}\mathbf{EG}_{CY_5}^{\chi=\pm 48}(\tau,z) &= \mp\Big[22\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)-2\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &\qquad+\sum_{n=1}^{\infty}A_n\left(\mathbf{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right)\Big]\\ &= \mp 2f_{1a}(\tau,z)\,,\end{aligned}\tag{6.40}$$

i.e., a function whose coefficients are all sums of irreducible representations of $\mathbf{M}_{24}$ without an overall prefactor. One would therefore have expected that the twined functions, regardless of whether they correspond to 2A of $\mathbf{M}_{24}$, have no additional prefactor either, i.e., we would have expected to find $\pm 2f_{2a}$. However, these twined elliptic genera are instead given by $\{-42f_{2a},-38f_{2a},-22f_{2a},-6f_{2a},50f_{2a}\}$, i.e. we never recover the 2A series of $\mathbf{M}_{24}$ exactly. This is an interesting result for another reason: The states counted with sign by the elliptic genus arise from a cancellation only if at a special point in moduli space there exist states with the same mass and charge but different statistics. In the absence of such cancellations the twined elliptic genus should never result in a larger number of states with a given mass and charge than the untwined case, since the twined case is a restriction of a count of states that are invariant under the twining action. However this is exactly what is observed in some cases. For example, consider the Calabi–Yau 5-fold described as a hypersurface in the weighted projective space $\mathbb{CP}^6_{16,17,17,34,58,62,102}$ whose elliptic genus is given by the equation (6.40)(with a plus sign). Upon twining by a $\mathbb{Z}_2$ symmetry that acts as

$$\mathbb{Z}_2:\begin{cases}\Phi_2\to-\Phi_2\\ \Phi_6\to-\Phi_6\end{cases},\tag{6.41}$$

we find the following twined elliptic genus

$$\begin{aligned}\mathbb{Z}^{tw}_{CY_5}(\tau,z) &= 150\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)-50\left(\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &\qquad-150\left(\mathbf{ch}^{\mathcal{N}=2}_{5,1,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,1,-\frac{3}{2}}(\tau,z)\right)+\mathcal{O}(q^2)\\ &= 50f_{2a}(\tau,z)\,.\end{aligned}\tag{6.42}$$

This means that the 22 states that multiply $\mathbf{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)$ in equation (6.40) now have a multiplicity of 150 under a $\mathbb{Z}_2$ twining as described above. The interpretation that we give here is that the 22 untwined states must likely arise from a cancellation between $n+22$ bosonic states and n fermionic states. Upon twining we have $m+150$ bosonic states and m fermionic states with the condition that $n+22>m+150$ since in the twined case we do not count all states but just the ones invariant under the twining. By comparing equations (6.40) and (6.42) we see that in this example there must have been such a cancellation of states at each level since the twined elliptic genus has larger expansion coefficients. This occurrence has also been observed for CY 5–folds with larger Euler characteristic [**BCK+18**]. This can probably be understood as follows: Generic CY 5–folds have several Hodge numbers that are large compared to 48. Therefore, in order to obtain a CY 5-fold with a small Euler number, these large numbers have to cancel rather precisely. In the case above we have the non–trivial Hodge numbers

$$h^{1,1}=25,\quad h^{1,2}=0,\quad h^{1,3}=232,\quad h^{1,4}=259,\quad h^{2,2}=1692,\quad h^{2,3}=1946.\tag{6.43}$$

For any Calabi–Yau 5-fold one has [Gri99]

$$\chi_0 = \chi_5 = 0, \quad \chi_1 = \chi_4 = -\frac{1}{24}\chi_{CY_5}, \quad \chi_2 = \chi_3 = \frac{11}{24}\chi_{CY_5}, \tag{6.44}$$

which is consistent with the Hodge numbers above and $\chi_{CY_5} = -48$. Upon twining, there is a removal states of that do not preserve the twining symmetry. This results in a less precise cancellation between the large Hodge numbers which accounts for the larger multiplicities of the twining genera.

REMARK 6.4.1. *Ambient weighted projective spaces with six complex dimensions are generically singular and the Calabi–Yau 5–folds that are hypersurfaces in these spaces are also singular. Therefore, a large number of Calabi–Yau hypersurfaces in weighted projective spaces that we studied consists of CY 5–folds that are generically singular. However, this by itself does not explain why there is no connection to the Mathieu group for any of these cases. For example, the case of $T^4/\mathbb{Z}_2$ is a singular limit of $K3$ and it nevertheless gives rise to many twined series that one expects from Mathieu moonshine.*

Lastly, we discuss the twining by order two elements for the Calabi–Yau manifolds with $\chi = \pm 24$ that we constructed and studied with up to fourteen $\mathbb{Z}_2$ twinings each. However, none of the explicit cases of twined elliptic genus are proportional to f_{2a}, but rather linear combinations of f_{1a} and f_{2a} with non-zero coefficients for both functions. Therefore, as a final result of this analysis: We may exclude the existence of a strict $\mathbf{M}_{24}$ or $\mathbf{M}_{12}$ symmetry for the relevant CY 5–folds $\chi = \pm 48$ and $\chi = \pm 24$ that we constructed [BCK+18].

6.5. A comment on toroidal orbifold and two Gepner models

In the previous section, we studied the elliptic genus of twined CY 5–folds with the hope of finding an explicitly CY 5-fold whose elliptic genus has an exact $\mathbf{M}_{24}$ symmetry. Despite in–depth analysis, no evidence that relates $\mathbf{M}_{24}$ whose irreducible representations appear in the expansion of the elliptic genus of the CY 5-fold has been found. These results suggest that the elliptic genus of generic CY 5–folds do not have symmetries that correspond to the $\mathbf{M}_{24}$ group. However, it is still possible that one or several special CY 5–folds have $\mathbf{M}_{24}$ as their symmetry group or that the appearance of the Mathieu moonshine is described in another setting [BHL+20]. This is something that is as of yet not falsifiable since there is no exhaustive list of CY 5–folds. We may however check special models that have large symmetry groups, such as toroidal orbifolds or Gepner models. The advantage here is that it is not only possible to compute the elliptic genus, but one can also calculate the Hodge-elliptic genus [KT17, Wen17] or the fully flavoured partition function. If these models have an actual $\mathbf{M}_{24}$ symmetry, then coefficients in the elliptic genus, the Hodge-elliptic genus and the full partition function should be sums of irreducible representations of $\mathbf{M}_{24}$. However, in the examples that are studied in [BCK+18], it is difficult to discern the exact connection to the $\mathbf{M}_{24}$ group due to these models having large coefficients in their expansions. An investigation of the study of sporadic group symmetries in toroidal orbifolds T^{10}/G and Gepner models $(1)^6$, $(2)^4$, and the relation between the Umbral groups and the elliptic genus of CY 6–folds has also been carried out in [BCK+18] by the author of this thesis, however, this final report does not expand upon it and we refer the reader to [BCK+18] for more details.

6.6. Conclusions

In this chapter, we investigated some of the properties of the Mathieu moonshine viz., if it is a property of the $K3$ sigma model or a Jacobi form that also appears in the elliptic genera of higher dimensional Calabi–Yau manifolds. The idea of this chapter was to study the symmetries of the twined elliptic genera of Calabi–Yau 5–folds which are hypersurfaces in weighted projective space. The reason for choosing the case of CY 5–folds is that their elliptic genus contains the function $\phi_{0,1}$ and could therefore be used to study the question if whether this function is serves as the key element in the study of Mathieu moonshine. These manifolds have the advantage that their BPS spectra localizes at special points/cycles in such a way that they can be computed using localization techniques as studied in [**BEHT14**]. By constructing a large class of CY 5–folds and studying their twined genera, we make interesting observations that most of the cases of twined elliptic genera do not exhibit any relation to $\mathbf{M}_{24}$. A small number of these manifolds however, despite being related to the twined elliptic genera of linear combinations of the 1A and 2A elements of the $\mathbf{M}_{24}$ group, fail to demonstrate a smoking gun relation to $\mathbf{M}_{24}$, which would mean that all twining genera of all orders would be related to only one conjugacy class of $\mathbf{M}_{24}$ that is determined by the twining element. While this does not falsify the possibility of a Mathieu moonshine in the moduli space of CY 5–folds, it does however lend more support to the idea that Mathieu moonshine is likely not a feature of the elliptic genus of $K3$ per se, but rather of the whole moduli space of $K3$ surfaces, with the most promising evidence in this direction coming from [**GTVW14**]. This only deepens the mystery of Mathieu moonshine and lends more support to the statement that $K3$ surfaces are central in understanding the Mathieu moonshine phenomenon.

As a concluding remark, moonshine phenomena are special and rare mathematical phenomena between (possibly) physical partition functions and sporadic groups. Their study, thus far having led to advancements in conformal field theory [**Bor92**], also could serve as a window into understanding some of the discrete symmetries of string theory. It remains to be seen what physical theory or object really admits an action of such sporadic groups.

Part 3

Automorphic forms and black holes

CHAPTER 7

Counting BPS black holes in string theory

Overview of this chapter

The study of thermodynamics of black holes in gravitational theories [**Bek73, BCH73, Wal95**] is of prime importance as it is one of the most important elements in constructing a quantum theory of gravity. The black hole entropy, also known as *Bekenstein–Hawking entropy*, is of the form (in natural units of $\hbar = c = \ell_p = 1$)

$$S_{BH} = \frac{A}{4G_N^{(d)}} + c_1 \log A + c_2 \frac{1}{A} + \cdots + c_n e^{-A} \,, \tag{7.1}$$

where A is the codimension–1 area of the black hole, $G_N^{(d)}$ is the d–dimensional Newton's constant. The leading contribution term $\frac{A}{4G_N^{(d)}}$, resulting in what is known as the *area law*, is a universal term regardless of the black hole. Black hole thermodynamics and the computation of entropy is of importance, in particular, to address the issue of unitarity of the gravitational S–matrix. The nature of considering a quantum field theory on a black hole geometry in the semi–classical limit leads to a phenomenon known as *Hawking radiation* which, in the semi–classical limit, leads to a loss of unitarity in the aforementioned quantum field theory [**Haw75, Haw76**]. While there have been many ways to remedy this (see [**Har16**] and references therein), there is still the issue of counting the number of microstates of a black hole i.e., formulate a quantum theory which counts the microscopic degrees of freedom of a black hole. Progress towards this in string theory/supergravity was initiated by the work of [**SV96**] where the leading order term to (7.1) was derived from D–brane/conformal field theoretic techniques. However, the key goal has always been to compute (7.1) exactly. This is known as *precision counting* of black hole microstates [**DDMP05, Sen08**]. The aim of this chapter is provide sufficient background to understand the idea of precision counting and *exact holography* [**DGM13**] (which can be thought of as precision counting from both the supergravity and SCFT sides of the holographic duality). The key goal is to connect to the work presented in Chapter 8 where a crucial aspect of exact microstate counting in $\mathcal{N} = 4,\ d = 4$ theories is explained.
For more pedagogical reviews, we refer the reader to a vast array of literature found in [**DN12, Sen08, DDMP05**] and references therein. We in particular refer the reader to these references since a the notation used in this thesis follows from there. For an introduction into the number theoretic aspects of black holes, we refer the reader to [**DMZ12, Moo07, Moo98, DM11, Bel08**]. Regarding black holes in supergravity, we refer the reader to [**Moh00, BFM06, Bel08**].

7.1. Relevant aspects of $\mathcal{N} = 4,\ d = 4$ string compactification

7.1.1. Moduli spaces and charge vectors. The starting point to construct $\mathcal{N} = 4,\ d = 4$ black holes in string theory is to consider a compactification to a theory with 16 supercharges. Such a compactification is feasible by studying Type II string theory on $K3 \times T^2$, which is dual to Heterotic string theory on $T^4 \times T^2 \cong T^6$ [**Wit95, HT95**]. For details of compactification on $K3$ surfaces, we refer the reader to [**Asp96, AM96**]. The compactification of string theory on $K3 \times T^2$ results in the supergravity multiplet, 22 vector massless multiplets. The massless fields parametrize and S– and T–moduli spaces of the theory. The S–moduli space is

$$\mathcal{M}_S = SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1)\,, \tag{7.2}$$

while the T–moduli space (also known as the *Narain moduli space*) is

$$\mathcal{M}_T = O(22,6,\mathbb{Z})\backslash O(22,6,\mathbb{R})/\left(O(22,\mathbb{R})\times O(6,\mathbb{R})\right). \tag{7.3}$$

The $SL(2,\mathbb{Z})$ forms the S–duality group while the group of discrete identifications $O(22,6,\mathbb{Z})$ forms the T–duality group. The full U–duality group is $SL(2,\mathbb{Z})\times O(22,6,\mathbb{Z})$. What will be of relevance in this thesis is the study of the rest frame supersymmetric representations of these fields. The spectrum of the theory is as follows:

(a) The non–BPS multiplet (256 dimensional).
(b) The long–BPS multiplet (64 dimensional) which preserves only a quarter of the supersymmetry and is therefore a $\frac{1}{4}$–BPS multiplet.
(c) The short–BPS multiplet (16 dimensional) which preserved only half the supersymmetry and is therefore a $\frac{1}{2}$–BPS multiplet.

These BPS states can be specified by their charges alone and these charges are specified by the vector representation of the $O(22,6,\mathbb{Z})$ T–duality group and in the fundamental representation of the $SL(2,\mathbb{Z})$ S–duality group. In other words, the BPS states can be thought of as vectors in a $\Lambda^{22,6}$ self–dual, integer, even charge lattice of signature $(22,6)$. These charge vectors $\Gamma \in \Lambda^{22,6}$ are dyonic i.e., they carry electric and magnetic charges as $\Gamma = \begin{pmatrix} P \\ Q \end{pmatrix}$ where P is the magnetic and Q is the electric vector, each of length 28 that transform as previously mentioned under the S– and T–duality groups. A charge vector $\Gamma = \begin{pmatrix} P \\ Q \end{pmatrix}$ is said to be primitive if it cannot be identified (up to integer multiplication) with another vector in the charge lattice. A purely electric charge vector is one in which $P = 0$, while a purely magnetic vector is one in which $Q = 0$. An important distinction between $\frac{1}{4}$–BPS and $\frac{1}{2}$–BPS vectors is that for a $\frac{1}{4}$–BPS vector $\begin{pmatrix} P \\ Q \end{pmatrix}$, $P \nparallel Q$ while for a $\frac{1}{2}$–BPS vector, $P \parallel Q$. The action of the S–duality group on a charge vector is

$$\begin{pmatrix} P \\ Q \end{pmatrix} \xrightarrow{SL(2,\mathbb{Z})} \begin{pmatrix} aP + bQ \\ cP + dQ \end{pmatrix},\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{7.4}$$

Naturally, the action of the S–duality group also changes the S–modulus of the theory. The case of the T–duality transforms are a bit more complicated. We know that the T–moduli can be written in terms of 28×28 matrices, owing to the structure of their moduli space (7.3). Let $L \in \mathrm{Mat}(22,6,\mathbb{Z})$ be given by

$$L = \begin{pmatrix} \mathbf{C}_{E8\times E_8} & 0 & 0 \\ 0 & 0 & \mathbb{I}_6 \\ 0 & \mathbb{I}_6 & 0 \end{pmatrix}, \tag{7.5}$$

where $\mathbf{C}_{E8\times E_8}$ is the Cartan matrix of $E_8 \times E_8$ and $\mathbb{I}_6$ is the 6 dimensional identity matrix. A T–duality transform is a given by

$$P \to TP,\ Q \to TQ\,, \tag{7.6}$$

where $T := T^t L T = L$. Naturally, the action of the T–duality also changes the action of the T–modulus. Since moving in the moduli space changes the moduli, it can also change the charges of the theory. There are, however, quantities that do not change as a result of T-transformations. These are the T–dual invariants and are given by

$$n = Q^t L Q = \frac{Q^2}{2},\quad m = P^t L P = \frac{P^2}{2},\quad \ell = Q^t L P = P^t L Q = Q\cdot P = P\cdot Q. \tag{7.7}$$

The above T–dual invariants can be combined to yield the discriminant of a charge vector as $\Delta = 4mn - \ell^2 = P^2 Q^2 - (P\cdot Q)^2$. The discriminant for the case of $\mathcal{N}=4,\ d=4$ compactification that we consider here is also invariant under S–duality transforms i.e., the discriminant of the charge vector does not change in the S–moduli space either. This implies that the Δ is a U–duality invariant [**DGM11a**]. In this thesis, we shall consider the case that P, Q are such that $\gcd(P\wedge Q) = 1$.[1] Dyons which obey this relation are referred to as torsion 1 dyons. Dyons for which $\gcd(P\wedge Q) > 1$ will not be studied in this thesis and we refer the reader to [**BSS08b, BSS08a**] for more details.

7.1.2. Making black holes in string theory. From the perspective of type IIB string theory, it is also possible for us to assign a brane construction of BPS objects. This is done by wrapping D1 and D5 branes on $K3\times T^2 \cong K3\times S^1\times \tilde{S}^1$. Such black hole solutions have been studied well [**HS91, CM96, Mal96, HS91**]. Essentially a dyonic black hole solution can be constructed by first compactifying on $K3\times S^1$ and then compactifying again on $\tilde{S}^1$. The charges of the black hole are sourced as follows:

(a) The electric charge $Q^2/2 = n$ is due to the momentum along the circle S^1 and the KK monopole charges that arise upon further compactification on $\tilde{S}^1$.
(b) The magnetic charge $P^2/2 = m = Q_1 Q_5$ arises from taking Q_1 D1 branes wrapping S^1 and Q_5 D5 branes wrapping $K3\times S^1$.
(c) The angular momentum $\ell = P\cdot Q$ arises from taking motion in the KK monopole background i.e., ℓ units of angular momentum along $\tilde{S}^1$.

In this thesis, we shall be interested in two particular types of black hole solutions viz., the $\frac{1}{2}$–BPS zero area black hole and $\frac{1}{4}$–BPS large black hole. These black hole solutions have different behaviours across the moduli space as we shall see later.

(a) **$\frac{1}{2}$–BPS Black holes:**
$\frac{1}{2}$–BPS black holes are zero area black holes (at zeroth order in α' [**DDMP05**]) and do not contribute to the area law term in the Bekenstein–Hawking entropy (7.1). They have no center of mass motion in the KK monopole background.
(b) **$\frac{1}{4}$–BPS Black holes:**
$\frac{1}{4}$–BPS black holes are "large black holes" and contribute to the area law. They are described by the D1–D5-KK-p system and include a motion in the KK–monopole background [**BMPV97**].

These black holes can also be distinguished and constructed on the basis of the $4d/5d$ uplift where $\frac{1}{2}$–BPS states lift to black strings in a Taub–NUT geometry in 5d, while $\frac{1}{4}$–BPS black holes lift to spinning black holes in a Taub–NUT geometry [**GSY06b, GSY06a**].

[1] $\gcd(P\wedge Q) = 1$ means that $P_i Q_j - P_j Q_i = 1,\ i,j = 1,\cdots,28$.

7.2. BPS states counts from supergravity: Attractors, Localization and Exact Holography

While there are BPS black hole solutions in string theory, there is a specific reason to study and count BPS spectra in non–geometric settings. This is due to the lack of dependence of the BPS spectrum on the coupling of the theory. In other words, the BPS partition function (which includes functions such as the elliptic genus (3.48)) do not depend on the string coupling. This means that the BPS partition function in the limit of large string coupling $g_s \gg 1$ where the gravitational back reaction creates a black hole solution is the same as in the limit of small string coupling $g_s \ll 1$ which is described by a weakly coupled SCFT. This means that the partition function of the BPS states in the SCFT is the same as the partition function of the black hole in the gravitational description of the theory, which leads to a holographic interpretation for the entropy of these black holes, in the spirit of [**AGM$^+$00**, **Mal99**]. This idea was used to show that the leading entropy computation in the SCFT matches with the area law for the D1–D5 black hole in a five dimensional compactification in [**SV96**] and has led to extensive work on microstate counting thereafter. We refer the reader to [**MS11**, **DM11**, **DDMP05**] for more details. In this thesis, we shall focus on the SCFT aspects of black hole counting, although it would behoove us to motivate the notion of exact holography, localization and the attractor mechanism for now. In simple terms, exact holography from the perspective of this thesis is the technique to obtain an exact match of the black hole partition function/entropy from the gravitational and field theoretic description to all orders of expansion beyond the area law [**DGM13**, **Gom11**].

7.3. A comment on localization of supergravity

In this section, we discuss the first ingredient in exact holography viz., localization of supergravity. Localization is a technique from differential geometry [**DH82**, **AB94**] that has been applied to the study of BPS states in field theory [**P$^+$17**] and supergravity [**DGM13**, **DGM11b**, **Rey16**]. We refer the reader to these papers and thesis the [**Rey16**] for more references. Localization for black holes in $\mathcal{N} = 4,\ d = 4$ supergravity will be reviewed in Section 8.3. The idea of exact holography would therefore be to obtain an exact match between the black hole degeneracies obtained via localization of $\mathcal{N} = 4$ supergravity and the BPS state count obtained from the BPS partition function from the SCFT description, for any and all charges.

7.4. The attractor mechanism

The attractor mechanism for black holes [**FKS95**, **FK96**] (see [**Moo98**, **Sen08**] for reviews) states that fluctuations to asymptotic moduli do not affect the black hole entropy and the entropy is fixed in terms of the attractor equations in moduli space.[2] While we will not go into the details of solving the attractor equations, we do wish to point out the reason of motivating them here:

(1) The attractor mechanism fixes the entropy of the black hole in terms of the charges of the black hole.
(2) If we consider a moduli space in which the degeneracy of the black holes are constant (such as in the case of $\frac{1}{2}$–BPS solutions in $\mathcal{N} = 4,\ d = 4$, as we shall see), the attractor mechanism is a sufficient description of the BPS counts

[2] Or a domain in moduli space where the black hole degeneracy will not change.

from supergravity. For the case of $\frac{1}{2}$–BPS black holes in $\mathcal{N}=4,\ d=4$, the attractor mechanism has bee studied in [**LCdWKM04**].

7.5. Black holes and automorphic forms

Following the work of [**SV96**], there have been many bodies of work in trying to count the BPS partition function for black holes from an SCFT point of view [**MMS99**, **DMVV97**, **DVV97**, **SSY06a**, **SSY06b**, **DGM11a**, **Mal96**, **Gai05**, **CV07**, **DGM11a**, **DMZ12**]. The key ingredient in the SCFT computation of BPS black hole entropy is that the degeneracy of the black hole in the sense of $W = e^{S}_{BH}$ are computed charge wise and are given in terms of the Fourier expansion coefficients of the partition function of the SCFT, which as we have already seen before in Chapter 2, are automorphic forms. Automorphic forms show up in a variety of black hole and moduli space related topics and also for other string compactifications for example in [**Moo98**] but we shall not cover them in this thesis. Our sole focus in this section will be to define/derive automorphic forms that capture the degeneracies of BPS black holes in $\mathcal{N}=4,\ d=4$ theories.

7.5.1. Modular forms and small black holes. Small black holes, as we have seen in Section 7.1.2, are $\frac{1}{2}$–BPS objects in the compactification that we are considering. The entropy of such black holes in in fact captured by the Dedekind–eta function as in (2.7b) as

$$Z_{1/2}(\tau) := \frac{1}{\eta(\tau)^{24}} = q^{-1}\prod_{i=1}^{\infty}\frac{1}{(1-q^i)^{24}}\,, \tag{7.8}$$

with a Fourier expansion given by

$$Z_{1/2}(\tau) = q^{-1} + 24 + 324\, q + 3200\, q^2 + \cdots\ . \tag{7.9}$$

The derivation of this modular form can be performed in the dual heterotic frame where these BPS states are part of the perturbative spectrum [**DH89**]. These are the *Dabholkar–Harvey* (DH) states and will play an important role in this thesis. The idea behind their derivation is as follows. Due to supersymmetry, it suffices to restrict ourselves to the purely bosonic sector which is the spectrum of the right movers in the ten dimensional heterotic string. Therefore, the partition function of $\frac{1}{2}$–BPS objects is the partition function of the right movers in the heterotic string theory, which yields the Dedekind–eta function (7.8). In [**LCdWKM04**], this degeneracy was reproduced for the $\frac{1}{2}$–BPS black holes following the techniques using the attractor mechanism.
The partition function $Z_{1/2}(\tau)$ admits the Fourier series as in (7.9) where the interpretation of these Fourier coefficients c_i is that they are the degeneracy of the $\frac{1}{2}$–BPS black hole of either electric or magnetic charge $2i = Q^2$ or P^2. A rule of thumb is that a modular parameter corresponds to either an electric or a magnetic charge, while the elliptic parameter corresponds to the motion in the KK monopole background and is therefore related to the angular momentum of the large black hole.
The partition function $Z_{1/2}(\tau)$ is constant at all points in moduli space implying that the count of $\frac{1}{2}$–BPS states does not change in the overall moduli space $\mathcal{M}_s \times \mathcal{M}_T$ as in (7.2) and (7.3). Furthermore, since $Z_{1/2}(\tau)$ is a weakly holomorphic modular form, it has an exponential growth of degeneracies.

7.5.2. Jacobi forms and single charge, large black holes. We have already encountered Jacobi forms in Section 2.2. In the present context, these functions describe three charge black holes with area, electric charge, and fixed magnetic charge.[3] These objects are related to the elliptic genera and indices of BPS states. From the point of view of black holes in $\mathcal{N}=4,\ d=4$ string theory, it is useful to keep in mind the elliptic genus of $K3$(which is an index 1 Jacobi form) defined in (3.50) with the Fourier expansion

$$\mathbf{EG}_{K3}(\tau,z) = 2\phi_{0,1}(\tau,z) = 2\sum_{n,\ell} \underbrace{C(4n-\ell^2)}_{C_{K3}(\Delta)} q^n y^\ell \ . \tag{7.10}$$

In the above equation (7.10), the coefficients $C(\Delta)$ are of prime importance.[4] We shall denote these coefficients as $C_{K3}(\Delta)$ for future reference. To make a connection to black holes, it is important to consider the Hilbert scheme of $K3$, Hilb$(K3)$. The Hilbert scheme is related to the symmetric product orbifolds of $K3$ as a resolution of the singularities in the symmetric product of $K3$, $\mathrm{Sym}(K3)^{m+1} = (\underbrace{K3\otimes K3\otimes\cdots\otimes K3}_{m+1})/S^m$. More specifically, this resolution is necessarily a crepant resolution meaning that it does not change the canonical class of the manifold [**DHVW85, VW94, QW02**]. We shall refer to these objects equivalently. The Hilbert scheme/symmetric product of $K3$ will play an important role in the full dyonic partition function. We define the generating function of the elliptic genus of the Hilbert scheme as

$$\begin{aligned}\hat{Z}(\tau,\sigma,z) &= \sum_{m>-1} \mathbf{EG}_{\mathrm{Hilb}(K3)}\, \underbrace{e^{2\pi i m\sigma}}_{p^m} \\ &= \frac{1}{p}\prod_{m\geq 0, n\geq 0, \ell} \left(1-q^n p^m y^\ell\right)^{-2C_{K3}(\Delta)}\,,\end{aligned} \tag{7.11}$$

where the second line of the definition is derived from an orbifold [**DMVV97**]. Considering this Hilbert scheme paves way to constructing the full dyonic partition function [**DMVV97, DVV97**].
Jacobi forms can also be thought of as being the partition function of a dyonic black hole where either the electric/magnetic chemical potential of the black hole is treated in the micro–canonical ensemble and the magnetic/chemical potential is treated in the grand canonical ensemble. In other words, in order to compute the degeneracy of a dyonic black hole of a fixed electric and magnetic charge, the number theoretic technique is to fix one of them (this corresponds to a Fourier–Jacobi expansion of a Siegel modular form) and then perform a Fourier expansion of an appropriate Jacobi form.[5] To count BPS states of a three charge black hole with (say) electric charge n, angular momentum ℓ, and fixed magnetic charge m, we employ a Jacobi form $\phi_m(\tau,z)$. This Jacobi form has a Fourier expansion $\phi_m(\tau,z) = \sum_{n,\ell} c_m(n,\ell)q^n y^\ell$, where the coefficients $c_m(n,\ell)$ capture the degeneracy of a black hole at fixed magnetic charge m, electric charge $n = Q^2/2$, and angular momentum $\ell = P\cdot Q$. By electric–magnetic duality,

[3] Hence, only one modular parameter.

[4] The discriminant $\Delta = 4mn-\ell^2$. However, $m=1$ due to the index of the Jacobi form being 1.

[5] This split implicitly breaks electric–magnetic duality but the overall computation of the degeneracy respects this duality.

we can also keep the electric charge fixed in which case the modular parameter of the Jacobi form would correspond to the magnetic charge.

7.5.3. Siegel modular forms and dyonic black holes. We can finally address the generating function for the degeneracies of dyonic black holes. We shall restrict ourselves to the un–orbifolded case of $\mathcal{N}=4,\ d=4$ theories, although a lot of the work that will be mentioned here has already been generalized to the cases to CHL orbifolds in [**DJS07**, **DJS06**, **DS06**, **PVZ17**, **BCHP17**, **FKN20**, **CNR20**]. The starting point is the *Dijkgraaf–Verlinde–Verlinde* (DVV) conjecture.

CONJECTURE 7.5.1 (DVV). *The generating function of **all** $\frac{1}{4}$–BPS states in $\mathcal{N}=4,\ d=4$ string compactification is given in terms of the unique Igusa cusp form of weight 10, $\Phi_{10}(\tau,\sigma,z)$, as*

$$Z_{1/4}(\tau,\sigma,z)=\frac{1}{\Phi_{10}(\tau,\sigma,z)}=(qyp)^{-1}\prod_{(m,n,\ell)>0}\left(1-q^ny^\ell p^m\right)^{-2C_{K3}(4mn-\ell^2)}, \tag{7.12}$$

where $(m,n,\ell)>0$ is the notation used to denote that $\ell\in\mathbb{Z}$ if $m\geq 0,\ n>0$ and $\ell<0$ if $m=n=0$ and $C_{K3}(\Delta)$ are the coefficients of the Fourier expansion of the elliptic genus of $K3$, as defined in (7.10).

The properties of Siegel modular forms have been discussed in Section 2.3. In this thesis, we consider working with the product representation of the Igusa cusp form (7.12) which is obtained by the multiplicative lift of the elliptic genus of $K3$ [**GN97**, **Kaw97**, **DJS06**]. This is related to $\mathrm{Hilb}(K3)$ (7.11) as

$$\frac{1}{\Phi_{10}(\tau,\sigma,z)}=\frac{\hat{Z}(\tau,\sigma,z)}{\phi_{10,1}(\tau,z)}, \tag{7.13}$$

where $\hat{Z}(\tau,\sigma,z)$ is as defined in (7.11). (7.12) can be used to compute the degeneracy of $\frac{1}{4}$–BPS black holes via

$$d(m,n,\ell)=(-1)^{\ell+1}\int_{\mathcal{C}}d\tau\ d\sigma\ dz\frac{e^{-i\pi(n\tau+m\sigma+2\ell z)}}{\Phi_{10}(\tau,\sigma,z)}. \tag{7.14}$$

The above quantity is related to the helicity supertrace index, B_6 [**Sen08**, **Sen11a**]. The subtlety here arises from the $\mathcal{C}$ contour dependence. This subtlety is due to the fact that the contour $\mathcal{C}$ could in principle encompass a double pole[6] of $\dfrac{1}{\Phi_{10}(\tau,\sigma,z)}$ in the Siegel upper half plane (2.23), and in doing so, the integral (7.14) picks up the contribution from the residues at this double pole [**CV07**]. This can physically be explained in terms of *wall crossing phenomena.*

7.6. Comments on wall crossing phenomena

Wall crossing was first studied in [**CV93**] as a phenomenon in which a quantity/index[7] is only piece wise constant on moduli space, but changes discontinuously at special real codimension 1 loci in moduli space known as *walls of marginal stability.*

[6] $\dfrac{1}{\Phi_{10}(\tau,\sigma,z)}$ is a meromorphic function with double poles at $z=0$ and $Sp(2,\mathbb{Z})$ images.

[7] Here, the quantity is the B_6 index.

Standard references for wall crossing include [**Cec10**, **Pio13**, **KS08**, **DMZ12**]. Wall crossing phenomena has two interpretations in this case:

(1) The SCFT interpretation [**Cec10**]: The long BPS multiplet as defined in Section 7.1.1 splits into two short BPS multiplets. This split necessarily implies that split preserves the mass, central charge Z and necessarily leaves the phase of the BPS states equal to the original state i.e.,

$$\begin{aligned} \text{Central charge conservation: } & Z = Z_1 + Z_2 \,. \\ \text{Mass conservation: } & |Z| = |Z_1| + |Z_2| \,. \\ \text{Phase equality: } & \arg Z = \arg Z_1 = \arg Z_2 \,. \end{aligned} \tag{7.15}$$

This also can be written in terms of the charge vectors as

$$\begin{pmatrix} P \\ Q \end{pmatrix} \to \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} \oplus \begin{pmatrix} P_2 \\ Q_2 \end{pmatrix} . \tag{7.16}$$

The discriminant of the original charge vector is still preserved in terms of the decay products. Further restrictions on the form of (7.16) will be imposed in Chapter 8. These restrictions arise from the fact that a bound state may yield different decay products at different walls of marginal stability. We have considered a $1 \to 2$ split i.e., a decay with two end products. The conditions (7.15) hold for any number of decay products, as is usually the difficulty with $\mathcal{N} = 2$ theories. In $\mathcal{N} = 4,\ d = 4$ theories, the wall crossing phenomenon is much more restricted in the sense that a $\frac{1}{4}$–BPS state necessarily decays only into a product of $\frac{1}{2}$–BPS states at codimension 1 walls of marginal stability.

(2) **Gravitational interpretation:** From (7.15), it should not be too difficult to guess the gravitational interpretation of wall crossing phenomena. The interpretation is that it is thermodynamically favourable for $\frac{1}{2}$–BPS black holes to form a bound state (multi–center black holes) that effectively is $\frac{1}{4}$–BPS (or vice–versa where a $\frac{1}{4}$–BPS bound state decays) at precisely those value of moduli that are defined by walls of marginal stability. The walls of marginal stability are therefore loci in the moduli space where the bound state of the $\left(\frac{1}{2} \oplus \frac{1}{2}\right)$-BPS black holes degenerate i.e., the bound state radius diverges to ∞ [**Den00**].

The interpretation of wall crossing phenomenon is as follows:

(a) The residue contribution to (7.14) when the contour encompasses a pole is precisely the change in degeneracy due to wall crossing [**CV07**].

(b) The degeneracy of $\frac{1}{4}$–BPS states is not constant across moduli space.

(c) $\dfrac{1}{\Phi_{10}(\tau, \sigma, z)}$ also counts bound states of $\frac{1}{2}$–BPS black holes.

7.7. Mock Jacobi forms and single center black holes

In this section, we present a very pedagogical review of obtaining single center $\frac{1}{4}$–BPS black hole degeneracies from (7.12). The rigourous derivation can be found in [**DMZ12**] and is essential to ensure that the results are indeed consistent. We shall however skip most of the technical details here. We recall a key statement that the generating function for *all* $\frac{1}{4}$–BPS states in $\mathcal{N} = 4,\ d = 4$ compactification is given by

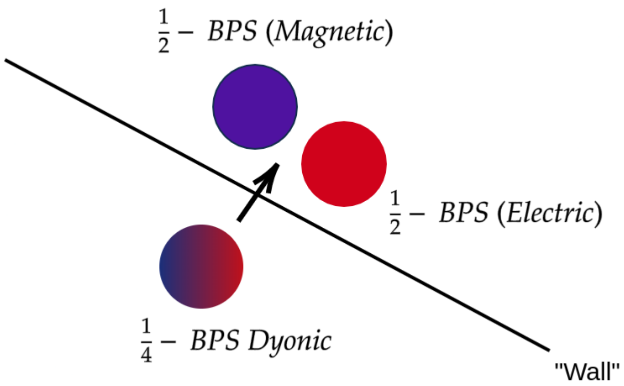


FIGURE 8. Wall crossing of the type $\frac{1}{4}$–BPS $\to$ $\frac{1}{2}$–BPS $\oplus$ $\frac{1}{2}$–BPS .

(7.12). We start with the Fourier–Jacobi expansion of (7.12) which gives us

$$\frac{1}{\Phi_{10}(\tau,\sigma,z)} = \sum_{m=-1}^{\infty} \psi_m(\tau,z)p^m, \tag{7.17}$$

where $\psi_m(\tau,z)$ are Jacobi forms of weight -10 and index m that are meromorphic along lines in the UHP. Note that the notion of double poles in the SUHP can be translated to lines (walls of marginal stability) in the UHP [**Sen07**]. These functions $\psi_m(\tau,z)$ are the generating function of *all* dyons of magnetic charge m. Note that the meromorphic variable z is still a contained in $\psi_m(\tau,z)$ and therefore the pole structure is still retained by this Jacobi form. In order to recover the single center $\frac{1}{4}$–BPS degeneracies for a state of given charge (m,n,ℓ), it does not simply suffice to know the Fourier expansion of (7.17) since there are also degeneracies of bound states captured by this function. We may however, perform a split of this Jacobi form into two pieces

$$\psi_m(\tau,z) = \psi_m^p(\tau,z) + \psi_m^F(\tau,z)\,, \tag{7.18}$$

where $\psi_m^P(\tau,z)$ denotes the 'polar' piece and is meromorphic and $\psi_m^F(\tau,z)$ is the finite piece and is 'holomorphic'. This preceding statement will be the focus of the next chapter. For now, we let us define the terms in (7.18). The starting point to compute $\psi_m^P(\tau,z)$ is to define an averaging function that averages over the double pole at $z=0$ and its $Sp(2,\mathbb{Z})$ images in the SUHP. This averaging function is given by

$$A_{2,m} = \mathrm{Av}^{(m)}\left[\frac{y}{(y-1)^2}\right], \tag{7.19}$$

where $A_{2,m}$ is the average over all double poles in the Jacobi form of index m, and Av is the standard averaging function defined as

$$\mathrm{Av}^{(m)}\left[\frac{y}{(y-1)^2}\right] = \sum_{\lambda} q^{m\lambda^2} y^{2m\lambda} \frac{q^{\lambda} y}{(y-1)^2}\,. \tag{7.20}$$

Properties of the averaging function can be found in §8 in [**DMZ12**]. The averaging function (7.20) can be used to derive the *Appell–Lerch sum* $\mathcal{A}_{2,m}$ as

$$\mathcal{A}_{2,m}(\tau,z) = \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s} y^{2ms+1}}{(1-q^s y)^2}. \tag{7.21}$$

Using the Appell–Lerch sum, we define the polar piece as

$$\psi_m^P(\tau,z) = \frac{p_{24}(m+1)}{\eta(\tau)^{24}} \mathcal{A}_{2,m}(\tau,z), \tag{7.22}$$

where $p_{24}(m+1)$ is the coefficient of q^m in the Fourier expansion of $\dfrac{1}{\eta(\tau)^{24}}$. This Appell–Lerch sum is precisely the function that exhibits wall crossing at the double poles of $z=0$ and its $Sp(2,\mathbb{Z})$ images in the SUHP [**DMZ12**]. Putting together (7.18) and (7.22), we have

$$\psi_m^F(\tau,z) = \psi_m(\tau,z) - \frac{p_{24}(m+1)}{\eta(\tau)^{24}} \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s} y^{2ms+1}}{(1-q^s y)^2}, \tag{7.23}$$

which is a holomorphic, mock–Jacobi form that is weakly holomorphic. Recall that this is required for the exponential growth of degeneracies. One would therefore expect that computing the Fourier expansion coefficients of (7.23) gives the degeneracies of $\frac{1}{4}$–BPS black holes $c_m^F(n,\ell)$ as

$$\psi_m^F(\tau,z) = \sum_{n,\ell} c_m^F(n,\ell)\; q^n y^{\ell}. \tag{7.24}$$

It is however challenging to compute $c_m^F(n,\ell)$ for arbitrary (m,n,ℓ). Therefore, we make use of the Rademacher expansion to compute these coefficients.

7.8. (Mock) Jacobi forms and the Rademacher expansion

In this section, we review the Rademacher expansion for a mock–Jacobi form following the introduction to the Rademacher expansion in Section 2.7 [**BO06, BM13, BM11, BO12**] in the current context of mock–Jacobi forms that arise as the generating function of single center black hole states in $\mathcal{N}=4$, $d=4$ string theory. The function that is relevant here is the mock Jacobi form ψ_m^F. In this case, the generalized Rademacher expansion for the Fourier coefficients $c_m^F(n,\ell)$, $\Delta \geq 0$, was obtained in

[**FR17**] and can be stated as

$$
\begin{aligned}
c_m^F(n,\ell) \;=\; & 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ 4m\widetilde{n}-\widetilde{\ell}^2<0}} c_m^F(\widetilde{n},\widetilde{\ell})\, \frac{\mathrm{KLS}\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m}\,;k,\psi\big)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\!\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right) \\
& + \sqrt{2m} \sum_{k=1}^{\infty} \frac{\mathrm{KLS}\big(\frac{\Delta}{4m},-1\,;k,\psi\big)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\!\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) \\
& - \frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g\in\mathbb{Z}/2mk\mathbb{Z} \\ g\equiv j (\mathrm{mod}\ 2m)}} \frac{\mathrm{KLS}\big(\frac{\Delta}{4m},-1-\frac{g^2}{4m}\,;k,\psi\big)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\
& \times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\!\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right) (1-mu^2)^{25/4}\, \mathrm{d}u\,,
\end{aligned}
\tag{7.25}
$$

where KLS is the generalized Kloosterman sum with the multiplier system $\psi(\gamma)$ given explicitly in [**FR17**], and the function $f_{k,g,m}$ is given by

$$
f_{k,g,m}(u) \;:=\; \begin{cases} \dfrac{\pi^2}{\sinh^2(\frac{\pi u}{k}-\frac{\pi i g}{2mk})} & \text{if } g\not\equiv 0\,(\mathrm{mod}\ 2mk)\,, \\[2ex] \dfrac{\pi^2}{\sinh^2(\frac{\pi u}{k})} - \dfrac{k^2}{u^2} & \text{if } g\equiv 0\,(\mathrm{mod}\ 2mk)\,. \end{cases}
\tag{7.26}
$$

The last two terms in (7.25) arise due to the mock modular nature of ψ_m^F. The key feature of (7.25) is that the coefficients $c_m^F(n,\ell)$ for $\Delta \geq 0$ (the left-hand side) are completely determined by the polar coefficients $c_m^F(n,\ell)$ for $\Delta < 0$, and the modular properties of ψ_m^F such as its weight, index, and multiplier system.

CHAPTER 8

Reconstructing mock–modular black hole entropy from $\frac{1}{2}$–BPS states

Overview of this chapter

This chapter is based on the results published in [**CKM+19**].

In the previous section, we inferred from (7.24) that the degeneracies of single–centered dyonic $\frac{1}{4}$–BPS black holes in Type II string theory on $K3 \times T^2$ are the coefficients of the Fourier expansion of certain mock Jacobi forms (7.23) obtained from the Igusa cusp form $\Phi_{10}(\tau, \sigma, z)^{-1}$ (7.12). These mock Jacobi forms were explained to be weakly holomorphic forms implying that their entropy is controlled by the polar states i.e., states of negative charge discriminant.
In this chapter we present an exact analytic formula for the degeneracies of the negative discriminant states in (7.23) purely in terms of the degeneracies of the perturbative $\frac{1}{2}$–BPS states, and due to the Rademacher circle method, show that the entropy of all single centered black hole states can be captured by $\frac{1}{2}$–BPS degeneracies of states.
We arrive at the formula by using the physical interpretation that these negative discriminant states are residual bound states in the finite mock Jacobi forms, and then making use of previous results in the literature to track the decay of such states into pairs of $\frac{1}{2}$–BPS states in the moduli space. Although there are an infinite number of such decays, we show that only a finite number of them contribute to the formula.
We also discuss in detail the phenomenon of BH bound state metamorphosis (BSM) [**Sen11b, ADJM12**]. We show that the dyonic BSM orbits with U–duality invariant $\Delta < 0$ are in exact correspondence with the solution sets of the Brahmagupta–Pell equation, which implies that they are isomorphic to the group of units in the order $\mathbb{Z}[\sqrt{|\Delta|}]$ in the real quadratic field $\mathbb{Q}(\sqrt{|\Delta|})$.
We present a reasonably large number of numerical checks of the exact formula in Appendix C against the numerical data from the Igusa cusp form to establish perfect agreement.

8.1. Motivation and set up of the problem

In the previous section, we mentioned that one of the aims of exact holography and precision counting is to ascertain a match between the calculations for the black hole degeneracy to all orders from the microscopic (SCFT) and macroscopic (supergravity) sides of the holographic duality. The motivation for this chapter arises from trying to obtain this exact match for the case of negative discriminant states in the mock Jacobi form $\psi^F(\tau, z)$ as in (7.23).

8.1.0.1. *Exact holography for $\mathcal{N} = 8,\ d = 4$ theories.* We first comment on the case of $\mathcal{N} = 8,\ d = 4$ compactifications. The $\frac{1}{8}$–BPS dyon spectrum can be computed by considering Type II on T^6 as being given by [MMS99]

$$Z_{1/8}^{\mathcal{N}=8}(\tau,z) = \phi_{-2,1}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}\,. \tag{8.1}$$

For this case, the exact degeneracies of supersymmetric black holes are given by the Fourier expansion coefficients of $Z_{1/8}^{\mathcal{N}=8}(\tau,z)$ (8.1) as

$$Z_{1/8}^{\mathcal{N}=8}(\tau,z) = \sum_{n,\ell} C_{\mathcal{N}=8}(n,\ell)\, e^{2\pi i n\tau}\, e^{2\pi i \ell z}\,. \tag{8.2}$$

The black hole is labeled by the discriminant $4n-\ell^2$ which grows, at large charges, as the square of the area of the horizon. The Hardy-Ramanujan-Rademacher formula for Jacobi forms provides an exact analytic expression for the coefficients $C_{\mathcal{N}=8}(n,\ell)$ of this Jacobi form as an infinite sum over Bessel functions with successively decreasing arguments. Importantly, this formula has no free parameters, and the only inputs are the modular transformation properties (the weight and index) of the Jacobi form $Z_{1/8}^{\mathcal{N}=8}$ and an overall fixed normalization.[1] Using localization techniques for supergravity theories, each term in the Rademacher expansion can be interpreted as a functional integral over the smooth fluctuations around certain asymptotically AdS_2 configurations [DGM11b, DGM13, DGM15].

8.1.0.2. *Exact holography and localization for $\mathcal{N}=4$, $d=4$ theories.* The next simplest examples to consider are $\frac{1}{4}$-BPS dyonic black holes in $\mathcal{N}=4$ string theories, which have been reviewed in Chapter 7. The main subtlety in $\mathcal{N}=4$ string theory compared to $\mathcal{N}=8$ string theory is that, at strong coupling, the supersymmetric index receives contributions from $\frac{1}{4}$-BPS single-centered black holes as well as bound states of two $\frac{1}{2}$-BPS black holes. The question then arises as to how to isolate the microstates that contribute to the single-centered black hole only and computing this requires us to abandon the modular symmetries of the Jacobi form and explain the single center black hole entropy through mock Jacobi forms. This has again been reviewed in Chapter 7. The degeneracies of the black holes are encoded in the Fourier expansion coefficients of these mock Jacobi forms and one may still employ the Rademacher technique to these mock–Jacobi forms [FR17] to obtain an exact analytic expression for the degeneracies of (7.24) and is given by (7.25). This means that the degeneracies of the single center $\frac{1}{4}$–BPS black holes are determined entirely by mock–modular properties of (7.18) and the polar terms therein. A *polar state* (and correspondingly *polar coefficient*) means a state (or coefficient thereof) with discriminant $\Delta = 4mn-\ell^2 < 0$. However, as we shall see, the input information to the Rademacher expansion which are the polar coefficients of the mock modular form themselves are not always in agreement with the corresponding degeneracies obtained from the gravitational localization calculation [MR16]. Therefore, to ascertain the exact count of these polar degeneracies is crucial for the utilization of the Rademacher technique.

In this chapter we present a simple analytic formula for the polar coefficients in terms of the degeneracies of $\frac{1}{2}$-BPS states in $\mathcal{N}=4$ string theory. In the heterotic duality frame, these are realized as perturbative fluctuations of the fundamental strings i.e., the Dabholkar-Harvey states [DH89]. This means that the full quantum degeneracy of the black hole—which is a non-perturbative bound state of strings, branes, and

[1]The Rademacher formula, as reviewed in Section 7.8, typically has a finite number of integers (the polar degeneracies) as input, but in this case the large symmetry of the theory implies there is only one independent polar degeneracy which can be normalized to one.

KK-monopoles—is completely controlled by simple perturbative elements of string theory. The nature of the formula, presented below in (8.5), first constructed in [**Sen11b**], is also important since it implies that the polar coefficients of the mock Jacobi forms are simply linear combinations of quadratic functions of the $\frac{1}{2}$-BPS degeneracies. The latter can be interpreted as counting worldsheet instantons or, more precisely, genus-one Gromov–Witten invariants. This structure is clearly reminiscent of the OSV formula $Z_{\rm BH} = |Z_{\rm top}|^2$ [**OSV04**], but the details are somewhat different. The right-hand side of (8.111), involving instanton degeneracies, is controlled by $Z_{\rm top}$, while the left-hand side is the "seed" for the $\frac{1}{4}$-BPS BH degeneracies via an intricate series which is dictated by the mock modular symmetry. The idea of exploiting modular symmetry in order to reach a precise non-perturbative definition for the OSV formula for $\mathcal{N} = 2$ theories was studied in [**DM11**], and demands a lot of further work since these theories are more complicated.

8.1.1. The main formula. We know thus far that the degeneracy of single center $\frac{1}{4}$–BPS black holes are controlled by the mock Jacobi form (7.23) with a Fourier expansion given by

$$\psi^F_m(\tau,z) \;=\; \sum_{n,\ell} c^F_m(n,\ell)\, e^{2\pi i n\tau}\, e^{2\pi i \ell z}\,. \tag{8.3}$$

The microscopic degeneracies of $\frac{1}{4}$-BPS single-centered black holes are related to these Fourier coefficients as

$$d^{\rm BH}_{\rm micro}(n,\ell,m) \;=\; (-1)^{\ell+1}\, c^F_m(n,\ell) \quad \text{for} \quad \Delta \;=\; 4mn-\ell^2 > 0\,. \tag{8.4}$$

We now present the analytic formula for the black hole degeneracies which is a combination of the following two formulas:

1. From the previous chapters, we know that the BH coefficients $c^F_m(n,\ell)$, $\Delta > 0$ are completely controlled by the polar coefficients $c^F_m(n,\ell)$, $\Delta < 0$. The relevant formula follows from the ideas of Hardy-Ramanujan-Rademacher applied to mock Jacobi forms, which by now has become a well-established technique in analytic number theory [**BO06, BM13, BO12, FR17**].
2. The polar coefficients $\widetilde{c}^F_m(n,\ell)$, $\Delta < 0$ required as input for the Rademacher expansion are given by

$$\widetilde{c}^F_m(n,\ell) \;=\; \sum_{\gamma\in {\rm W}(n,\ell,m)} (-1)^{\ell_\gamma+1}\, |\ell_\gamma|\, d(m_\gamma)\, d(n_\gamma) \quad \text{for} \quad \Delta \;=\; 4mn-\ell^2 < 0\,. \tag{8.5}$$

   We obtain this formula using the ideas and results of [**Sen11b**], by tracking all possible ways that a two-centered black hole bound state of total charge (n,ℓ,m) decays into its constituents across a wall of marginal stability. Here ${\rm W}(n,\ell,m)$ is a set of $\gamma \in SL(2,\mathbb{Z})$ matrices that encode the relevant walls of marginal stability, as studied in [**Sen07**]. This set is finite and we characterize this set completely. The precise formulas are given in (8.108), (8.111). The quantities $(n_\gamma,\ell_\gamma,m_\gamma)$ are the T-duality invariants of the charges (Q,P) transformed by γ, and $d(n)$ is the degeneracy of $\frac{1}{2}$-BPS states with charge invariant n, given by [**DH89**]

$$\frac{1}{\eta(\tau)^{24}} \;=\; \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i n\tau}\,. \tag{8.6}$$

8.1.2. The idea of the calculation. The key idea for this chapter is to interpret the polar terms of (7.23) as bound states of two $\frac{1}{2}$–BPS states. When the charges have a negative discriminant they cannot form a single-centered black hole since $\mathrm{A} \sim \sqrt{\Delta}$. We know that the only other configurations that contribute to the $\frac{1}{4}$-BPS index in $\mathcal{N} = 4$ string theory are two-centered bound states of $\frac{1}{2}$-BPS black holes [**DGMN10, Sen11b**]. Thus the problem becomes one of calculating all possible ways a given set of charges with negative discriminant contributing to c^F_m can be represented as two-centered black hole bound states.

Any such bound state is an S-duality ($SL(2,\mathbb{Z})$) transformation of the basic bound state, which consists of an electrically charged $\frac{1}{2}$-BPS black hole with invariant $n = Q^2/2$, a magnetically charged $\frac{1}{2}$-BPS black hole with invariant $m = P^2/2$, and the electromagnetic fields carry angular momentum $\ell = Q \cdot P$. The indexed degeneracy of this system equals [**Sen11b**]

$$(-1)^{\ell+1}\,|\ell| \cdot d(n) \cdot d(m)\,. \tag{8.7}$$

The factors $d(n)$ and $d(m)$ in this formula are, respectively, the internal degeneracies of the electric and magnetic $\frac{1}{2}$-BPS black holes, and the factor $(-1)^{\ell+1}|\ell|$ is the indexed number of supersymmetric ground states of the quantum mechanics of the relative motion between the two centers [**Den00**]. The degeneracy of an arbitrary bound state can be calculated by acting on the charges (Q, P) by the appropriate S-duality transformation and replacing the charge invariants in (8.7) by their transformed versions. This is precisely the structure of the formula (8.5).

The final ingredient of the formula is to state precisely what are the allowed values of γ which labels all possible bound states. A very closely related problem was solved in an elegant manner in [**Sen11b**], which we use after making small modifications. The basic intuition comes from particle physics—any bound state must decay into its fundamental constituents somewhere, and so the question of which bound states exist is the same as the question of what are all the possible decays of two-centered $\frac{1}{2}$-BPS black holes. As was shown in [**Sen11b**] the possible decays are labeled by a certain set of $SL(2,\mathbb{Z})$ matrices. The exact nature of this set is a little subtle due to a phenomenon called black hole bound state metamorphosis (BSM) [**ADJM12, Sen11b, CLSS13**], which identifies different-looking physical configurations with each other. This is the step which lacks a rigorous mathematical proof, but the physical picture is well-supported. The sum over $\mathrm{W}(n, \ell, m)$ in (8.5) is precisely the sum over all possible decay channels after taking metamorphosis into account. Thus the performed checks of the formula (8.5) can be thought of as providing more evidence for the phenomenon of metamorphosis.

The metamorphosis can be of three types: electric, magnetic, and dyonic. The corresponding identifications generate orbits of length 2 in the first two cases and of infinite length in the third. In the first two cases the metamorphosis has a simple $\mathbb{Z}/2\mathbb{Z}$ structure, while the group structure of the dyonic case was less clear so far. We show in this chapter that the identifications due to dyonic BSM have a group structure of $\mathbb{Z}$. Moreover, the problem of finding BSM orbits maps precisely to finding the solutions to a well-studied Diophantine equation, namely the Brahmagupta–Pell equation, whose structure is completely known. In the language of algebraic number theory, this is the problem of finding the group of units in the order $\mathbb{Z}[\sqrt{|\Delta|}]$ in the real quadratic field $\mathbb{Q}(\sqrt{|\Delta|})$.

8.1.3. Gravitational interpretation. Localization in supergravity allows us to study the quantum entropy of the gravitational theory is formulated as a functional integral over asymptotically AdS_2 configurations [**DGM11b, DGM13, DGM15**]. The result takes the form of an infinite sum of finite-dimensional integrals over the (off-shell) fluctuations of the scalar fields around the attractor background, where the integrand includes a tree-level and a one-loop factor in the off-shell theory. The infinite sum is interpreted as different orbifold configurations in string theory with the same AdS_2 boundary [**DGM15**]. In the $\mathcal{N} = 8$ theory, this result agrees exactly with the Rademacher expansion for the coefficient of the Jacobi form controlling the microscopic index.

We can now offer a physical interpretation of the novel exact degeneracy formula (7.25), (8.5) from this point of view. The sum over k in (7.25) runs over all positive integers with the argument of the Bessel function suppressed as $1/k$ and the Kloosterman sum depending on k. This part of the structure comes from a sum over $\Gamma_\infty\backslash SL(2,\mathbb{Z})$ of the circle method, and can be interpreted in the gravitational theory exactly as in the $\mathcal{N} = 8$ theory, namely as a sum over orbifolds of the type $(\mathrm{AdS}_2 \times S^1 \times S^2)/\mathbb{Z}_k$ [**DGM15**]. The Kloosterman sum arises from an analysis of Chern–Simons terms in the full geometry. The degeneracies of polar states $\widetilde{c}_m(n,\ell)$

$$\widetilde{c}_m(n,\ell) \;:=\; c^F_m(n,\ell) \quad \text{for} \quad \Delta = 4mn - \ell^2 < 0 \tag{8.8}$$

is interpreted as the number of states of a given (n,ℓ,m) which do *not* form a big single-centered BH. The finite sum over $\gamma \in \mathrm{W}(n,\ell,m)$ in (8.5) is indicative of a further fine structure where the smallest units are the $\frac{1}{2}$-BPS instanton states with their corresponding degeneracy. This is the sense in which the final degeneracy formula is constructed out of the worldsheet instantons.

Outline of chapter. The outline of the chapter is as follows: In Section 8.2, we review some of the concepts regarding the contour over which the Igusa cusp form can be integrated, and explain the structure of the attractor region and moduli space. In Section 8.3 we discuss the macroscopic supergravity counting of $\frac{1}{4}$–BPS states. Section 8.4 discusses several details that are important for the proper counting of negative discriminant states. In Sections 8.5 and 8.6 we present all the relevant calculations that will lead to the explicit formula for the negative discriminant states. In Section 8.6, we characterize the orbits of dyonic metamorphosis in terms of the orbits of the solutions to the Brahmagupta–Pell equation. The final Section 8.7 presents the exact black hole formula and Appendix C presents numerical data for the analysis presented in this chapter.

8.2. The moduli space and the attractor region

Having reviewed the $\frac{1}{4}$–BPS single-centered degeneracies in Chapter 7, we now discuss in a bit more detail the structure of walls in the moduli space. The moduli-dependent contour $\mathcal{C}$ in (7.14) can be written in terms of the moduli-dependent central charge matrix $\mathcal{Z}$ [**CV07**]. The latter can be parameterized[2] by a complex scalar $\Sigma = \Sigma_1 + \mathrm{i}\Sigma_2$ as

$$\mathcal{Z} \;=\; \Sigma_2^{-1} \begin{pmatrix} |\Sigma|^2 & \Sigma_1 \\ \Sigma_1 & 1 \end{pmatrix} . \tag{8.9}$$

[2]This parametrization corresponds to a projection from the full moduli space to the two-dimensional axion–dilaton moduli space of the heterotic frame [**Sen07**].

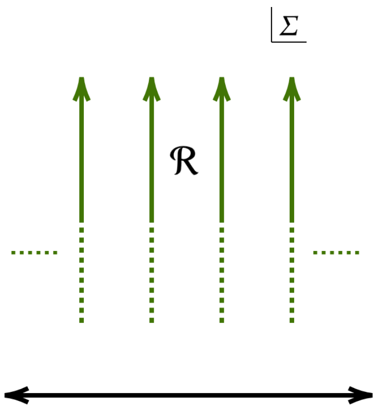
In terms of this matrix, the contour in (7.14) reads (with $\varepsilon \to 0^+$)

$$\begin{aligned}\mathcal{C} =&\big\{\mathrm{Im}(\tau) = \Sigma_2^{-1}/\varepsilon,\ \mathrm{Im}(\sigma) = \Sigma_2^{-1}|\Sigma|^2/\varepsilon,\\ &\mathrm{Im}(z) = -\Sigma_2^{-1}\Sigma_1/\varepsilon,\ 0 \le \mathrm{Re}(\tau), \mathrm{Re}(\sigma), \mathrm{Re}(z) < 1\big\}.\end{aligned} \tag{8.10}$$

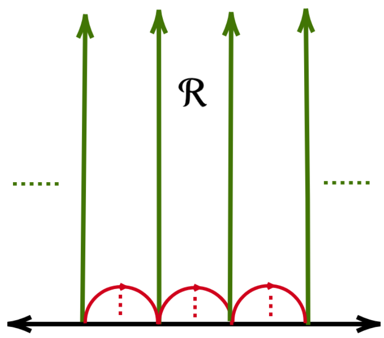
For the case of the mock Jacobi forms that we are considering, the attractor contour reduces to

$$\mathrm{Im}(z)/\mathrm{Im}(\tau) = -\ell/2m. \tag{8.11}$$

The Fourier–Jacobi expansion (7.17) then corresponds to taking the limit $\Sigma_2 \to \infty$ while keeping Σ_1 and $\varepsilon\,\Sigma_2$ fixed. This limit has a physical interpretation as the M-theory limit, where one of the circles inside the internal T^2 of the Type II frame becomes large [**DMZ12**]. In this limit, the expansion (7.17) around $\sigma \to i\infty$ takes us high into the upper half-plane parameterized by Σ, and varying Σ_1 moves us horizontally. This is depicted in Figure 9a. The wall–crossing captured, in the M–theory limit, by the



(A) The M–theory limit (7.17)



(B) The full Σ moduli space

FIGURE 9. The region $\mathcal{R}$ in the moduli space

Appell–Lerch sum (7.21) divides the moduli space into chambers separated by parallel marginal stability walls located at $\Sigma_1 = \alpha \in \mathbb{Z}$. The attractor contour (8.11) then corresponds to picking a particular chamber, which we denote by $\mathcal{R}$. As mentioned below (8.11), ℓ can be restricted to $0 \le \ell < 2m$, so it follows that $\mathcal{R}$ is the chamber between the walls located at $\alpha = 0$ and $\alpha = -1$. Thus,

$$\mathcal{R}\ :\ -\mathrm{Im}(\tau) < \mathrm{Im}(z) \le 0\,. \tag{8.12}$$

Now, in the chamber $\mathcal{R}$, the Fourier coefficients of ψ_m^{P} in the range $0 \le \ell < 2m$ vanish because the coefficients of the Appell–Lerch sum vanish in this chamber, as can easily be checked. Therefore, the Fourier coefficients of the meromorphic Jacobi forms ψ_m in the chamber $\mathcal{R}$ are equal to the coefficients of the finite part ψ_m^F. For $\Delta > 0$, these coefficients correspond precisely to the single–centered black hole degeneracies, as given by (8.4). However, ψ_m^F also has coefficients with $\Delta < 0$ which live in the chamber $\mathcal{R}$. Thus, we arrive at the following physical interpretation of the Fourier coefficients c_m^F: they count the indexed number of $\frac{1}{4}$–BPS dyonic states in the chamber $\mathcal{R}$. This is true for $\Delta > 0$ (which are single–centered black holes) as well as, importantly, for $\Delta < 0$ (which correspond to multi–centered black holes).

Because of the Rademacher formula (7.25), our task of finding an analytic formula for the single–centered degeneracies (8.4) is thus reduced to finding an analytic formula for the negative discriminant degeneracies, for which we will use the above physical picture. As explained in the introduction, the $\frac{1}{4}$–BPS bound states counted by (8.8) always decay upon crossing a wall of marginal stability. We therefore have to track them in the Σ moduli space and reconstruct them as a sum of their $\frac{1}{2}$–BPS constituents. The decays corresponding to moving horizontally by varying Σ_1 have been taken into account by ψ_m^{P}, so they will not contribute to $c_m^F(\Delta < 0)$. As discussed above, the contribution from ψ_m^{P} in the region $\mathcal{R}$ actually vanishes.[3] This is consistent with the analysis of [**Sen11b**]. In addition, there can be other decays in the full moduli space, and to see them we need to go away from the M–theory limit. As shown in [**Sen11b**], the region $\mathcal{R}$ extends also vertically downwards back to the $\Sigma_2 = 0$ line. Therefore, away from the M–theory limit, the negative discriminant states contained in $\mathcal{R}$ can also decay further upon crossing circular walls (the precise shape of these walls is charge–dependent), which are shown in Figure 9b. We can now use the results of [**Sen11b**] to count how many negative discriminant states live in the region $\mathcal{R}$ and obtain the polar coefficients (8.8). This will be reviewed in Section 8.4.

8.3. Localization of $\mathcal{N} = 4$ supergravity and black hole degeneracy

Before presenting the derivation of the formula for the polar coefficients $\widetilde{c}_m(n, \ell)$ defined in (8.8), we review how the main idea originates from physical considerations. In [**MR16**] the asymptotic degeneracies of $\frac{1}{4}$–BPS single–centered BHs as a supergravity functional integral in the AdS_2 near–horizon geometry of the BHs, following the ideas of [**Sen09, DGM11b**] were computed. This computation did not yield the exact answer matching the microscopic prediction. However, the results of [**MR16**] can be interpreted as an approximate relation between the polar coefficients of the counting function ψ_m^F and the Fourier coefficients of the Dedekind eta function, which was checked to be true to a good approximation. As we explain below, the main formula of the present chapter (8.5) can be seen as correcting the approximate result of [**MR16**] to an exact formula.

8.3.1. The quantum entropy of $\frac{1}{4}$–BPS single–centered black holes. While we mainly focus on the microscopic picture in the rest of the chapter, the origins of (8.111) are from a macroscopic intuition that we now review. Using ideas of the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence, a macroscopic supergravity description for the degeneracies of microstates of supersymmetric BHs, called the quantum entropy formalism, was put forward in [**Sen09**]. The near–horizon geometry of extremal black holes universally contains an AdS_2 factor, and the proposal of [**Sen09**] is that the degeneracies of supersymmetric extremal black holes is a functional integral on this AdS_2 space defined as

$$d_{\mathrm{macro}}(Q, P) \;=\; \Big\langle \exp\Big[q_I \int_{S^1} A^I \Big] \Big\rangle_{\mathrm{EAdS}_2}^{\mathrm{finite}}, \tag{8.13}$$

where the computation involves the expectation value of the Wilson line around the Euclidean time circle S^1 for the $U(1)$ gauge fields A^I under which the black hole is charged, q_I denotes the corresponding charges of the BH, and the superscript "finite"

[3] If we are however interested in another chamber of the moduli space where the Appell–Lerch sum does not vanish, it is important to remove the associated decays taking place when varying Σ_1.

indicates a particular infrared regularization scheme to deal with the infinite volume of the EAdS_2 factor in the near–horizon geometry. We refer the reader to [**Sen09**] for details.

To compute the path integral (8.13) beyond the leading large–charge approximation, powerful techniques of supersymmetric localization have been employed starting with the work of [**DGM11b**]. Localization has been an invaluable tool in the study of partition functions in gauge theories and in many cases has allowed us to reduce a complicated path integral to a much simpler *finite* dimensional integral. A complete review falls outside the scope of the present chapter,[4] so we will simply give the final result obtained in [**DGM13, GM13, MR13, MR15, GIJ15, dWMR18, JM19**] for (8.13) after localization and for the class of black holes we are interested in. For $\frac{1}{2}$–BPS black hole solutions of an $\mathcal{N}=2$ supergravity theory with holomorphic function F, we have (8.14)

$$d_{\mathrm{macro}}(Q,P) \;=\; \int_{\mathcal{M}_{\mathcal{Q}}} \prod_{I=0}^{n_V} \mathrm{d}\phi^I\,\mu(\phi^I)\,\exp\Big[4\pi\,\mathrm{Im}[F(p^I,\phi^I)] - \pi\,q_I\phi^I\Big]\,\big(\chi_{\mathrm{V}}(p^I,\phi^I)\big)^{2-\frac{1}{12}(n_V+1)}\,.$$

The definition of the various quantities entering (8.14) are as follows. The integral is over the manifold $\mathcal{M}_{\mathcal{Q}}$, which is characterized by the bosonic field configurations that are supersymmetric with respect to a specific supercharge $\mathcal{Q}$ preserved by the black hole solution. This manifold is (n_V+1)–dimensional, where n_V is the number of Abelian vector multiplets under which the black hole is charged, and ϕ^I denote the coordinates on $\mathcal{M}_{\mathcal{Q}}$. Unlike the case of localization of a generic $\mathcal{N}=2$ supergravity theory [**MR15**], the localization of $\frac{1}{4}$–BPS solutions in $\mathcal{N}=4$ supergravity demands a truncation to $\mathcal{N}=2$ and this truncation discards the hypermultiplet contribution from (8.14) [**MR16, SY06**]. The integrand is completely specified by the function $F(p^I,\phi^I)$ of the theory, which is a homogeneous holomorphic function of its arguments. The associated Kähler potential $\chi_{\mathrm{V}}(p^I,\phi^I)$ is built out of this function. Finally, we have denoted by $\mu(\phi^I)$ the measure on $\mathcal{M}_{\mathcal{Q}}$, which was not obtained from first principles in the above references, but constrained to be a function that contributes $O(1)$ growth to the entropy when all the charges are scaled to be large.

To apply the formula (8.14) to the $\frac{1}{4}$–BPS single–centered black hole solution of the $\mathcal{N}=4$ theory discussed in Chapter 7, one consistently truncates the latter theory to an $\mathcal{N}=2$ theory with $n_V=23$ multiplets and function [**SY06**]

$$(8.15)\quad F(p^I,\phi^I) \;=\; -\frac{X^1}{X^0}\,X^a\,C_{ab}\,X^b + \frac{1}{2\mathrm{i}\pi}\log\Big[\eta^{24}\Big(\frac{X^1}{X^0}\Big)\Big]\,, \quad \text{with} \quad X^I=\phi^I+\mathrm{i}\,p^I\,,$$

where C_{ab} is the intersection matrix on the middle homology of the internal $K3$ manifold and $a,b=2,\dots,23$. Using this data and assuming a certain measure on $\mathcal{M}_{\mathcal{Q}}$, [**MR16, Gom17**] showed that the finite dimensional integral (8.14) could be put in the form of a sum of I–Bessel functions indicative of a Rademacher–type expansion for the macroscopic degeneracies of single–centered $\frac{1}{4}$–BPS black holes, similar to the exact microscopic formula (7.25). The above assumption about the measure was essentially a statement of consistency with a certain way of expanding the microscopic formula (7.14), as we now explain.

8.3.2. The measure and Rational Quadratic Divisors. The z–integral in the microscopic degeneracy formula (7.14) can be performed by calculating residues at the so called Rational Quadratic Divisors (RQDs) of the Igusa cusp form Φ_{10}. The

[4]See [**P$^+$17**] for an introduction and reviews in the context of field theory.

leading contribution to the integral comes from the RQD located at $z=0$ [DVV97]. Near $z=0$, the Igusa cusp form behaves as

$$\Phi_{10}(\tau,\sigma,z) \;=\; 4\pi^2\,(2z-\tau-\sigma)^{10}\,z^2\,\eta(\tau)^{24}\,\eta(\sigma)^{24}+\mathcal{O}(z^4)\,. \tag{8.16}$$

The remaining integral in τ and σ can then be expressed as [DS06]

$$d_{\frac{1}{4}}(Q,P) \;\simeq\; (-1)^{\ell+1}\int_{\mathcal{C}_2}\frac{\mathrm{d}^2\tau}{{\tau_2}^2}\,e^{-\mathcal{F}(\tau_1,\tau_2)}\,, \tag{8.17}$$

where $\simeq$ indicates that there are subleading contributions coming from other RQDs (additional poles in Φ_{10}^{-1}), and $\tau=\tau_1+\mathrm{i}\tau_2$. The function $\mathcal{F}(\tau_1,\tau_2)$ is given by

$$\begin{aligned}\mathcal{F}(\tau_1,\tau_2) = \;& -\frac{\pi}{\tau_2}\left(n-\ell\tau_1+m(\tau_1^2+\tau_2^2)\right)+\ln\eta^{24}(\tau_1+\mathrm{i}\tau_2)+\ln\eta^{24}(-\tau_1+\mathrm{i}\tau_2)+12\ln(2\tau_2)\\ & -\ln\left[\frac{1}{4\pi}\left\{26+\frac{2\pi}{\tau_2}(n-\ell\tau_1+m(\tau_1^2+\tau_2^2))\right\}\right]\,,\end{aligned} \tag{8.18}$$

and the contour of integration $\mathcal{C}_2$ is required to pass through the saddle–point of $\mathcal{F}(\tau_1,\tau_2)$. This way of manipulating the microscopic degeneracy formula corresponds, in physics, to calculating the degeneracies of BHs whose magnetic as well as electric charges grow at the same rate. In contrast, the expansion studied in (7.17) in the previous chapter corresponds to fixing the magnetic charges and letting the electric charges grow.

Adding a total derivative term and comparing to the macroscopic localized integral (8.14), the authors of [MR16, Gom17] concluded that the measure factor, corresponding to the leading RQD of Φ_{10} located at $z=0$, should take the form

$$\mu(\phi^I) \;=\; m+E_2\Big(\frac{\phi^1+\mathrm{i}p^1}{\phi^0}\Big)+E_2\Big(-\frac{\phi^1-\mathrm{i}p^1}{\phi^0}\Big)\,, \tag{8.19}$$

where E_2 is the Eisenstein series of weight 2, related to the Dedekind eta function as

$$E_2(\tau) \;=\; \frac{1}{2\pi\mathrm{i}}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\log\eta(\tau)^{24}\,. \tag{8.20}$$

Using the measure (8.19) in the integral (8.14) leads to an infinite sum of I-Bessel functions coming from integrating term-by-term the series expansions of the prepotential and the measure [MR16]. It was noticed in [MR16] that this infinite sum begins with terms that become smaller up to a point, but that the integrals start diverging after a while. This behavior is characteristic of an asymptotic series, which prompted [MR16] to truncate the sum after a finite number of terms. This was achieved using a contour prescription given in [Gom17]. The end result, after evaluating the integrals, was then

$$\begin{aligned}d_{\mathrm{macro}}(Q,P) \;\simeq\; 2\pi\sum_{\substack{0\le\tilde\ell\le m\\ \tilde\Delta<0}}(\tilde\ell-2\,\tilde n)\,d(m+\tilde n-\tilde\ell)\,d(\tilde n)\,\frac{\cos\big(\pi(m-\tilde\ell)\ell/m\big)}{\sqrt{m}}\;\times\\ \times\;\Big(\frac{|\tilde\Delta|}{\Delta}\Big)^{23/4}\,I_{23/2}\Big(\frac{\pi}{m}\sqrt{|\tilde\Delta|\Delta}\Big)\,,\end{aligned} \tag{8.21}$$

where $d(n)$ is the n^{th} Fourier coefficient of the Dedekind eta function as given in (8.6), Δ is the usual discriminant $4mn-\ell^2$, and the I-Bessel function is defined in (2.48). Comparing the above macroscopic result to the Fourier coefficients (2.47) and (2.50), we see that $d_{\rm macro}$ is the first ($k=1$) term in the Rademacher expansion for a Jacobi form of weight -10 upon making the identification

$$c_m(\tilde n,\tilde\ell) \;=\; (\tilde\ell - 2\,\tilde n)\, d(m+\tilde n-\tilde\ell)\, d(\tilde n) \quad \text{for} \quad \widetilde\Delta = 4m\tilde n - \tilde\ell^{\,2} < 0\,. \tag{8.22}$$

This proposal, motivated by the exact computation of a supergravity path integral, already offered a very good numerical agreement with the microscopic data and hinted at an intricate relationship between the Fourier coefficients of a simple modular form (the Dedekind–eta function) and those of the more complicated mock Jacobi forms $\psi_m^{\rm F}$. However, detailed numerical investigations also showed that the formula (8.22) cannot be the complete answer, as evidenced by the small discrepancies between the left- and right-hand sides highlighted in the tables of [MR16].

Since the derivation reviewed above relied on the approximations related to the asymptotic nature of the series, it was already clear that (8.21) is just the beginning of the complete formula. In the rest of the chapter, we obtain the correct and exact relationship between the polar terms of $\psi_m^{\rm F}$ and the Fourier coefficients of $\eta(\tau)^{-24}$ based on a precise analysis of negative discriminant states in $\mathcal{N}=4$ string theory, as summarized in (8.5). Therefore, the results of the present chapter can be interpreted as giving us the precise way to take into account the subleading RQDs that correct the measure (8.19), and truncate the infinite sum of Bessel functions arising from (8.14).[5] [6]

8.4. Negative discriminant states and walls of marginal stability

Now we turn back to the main goal, which is to obtain an analytic formula for the degeneracies of negative discriminant $\frac{1}{4}$–BPS states $\widetilde{c}_m(n,\ell)$ as defined in (8.8) in terms of the coefficients of the Dedekind eta function. In this section we set up the problem in a convenient form after reviewing some facts about negative discriminant states and walls of marginal stability associated to negative discriminant state decays. As reviewed in the previous section, we are interested in counting the number of negative discriminant states in the region $\mathcal{R}$, which correspond to bound states of two $\frac{1}{2}$–BPS states. Following [Sen11b], a convenient way to do so is to count how bound states appear or decay as we move around the moduli space parameterized by Σ discussed around (8.9). When we cross a wall of marginal stability, bound states appear or decay and contribute to the degeneracies of all negative discriminant states contained in ψ_m^F. We now review the structure of these walls of marginal stability, referring the reader to [Sen07, Sen11b] for more details.

8.4.1. Walls of marginal stability: Notation.

(1) In the Σ upper half–plane, the walls of marginal stability are of two types [Sen07]:
   (a) Semi–circles connecting two rational points p/r and q/s such that $ps - qr = 1$. We denote these walls as *S–walls*.

[5] The idea of summing up the contributions from all the RQDs of Φ_{10} to obtain the exact degeneracy of the dyonic BH was put forward in [MP09], but the lack of good technology at the time also led to divergent sums.

[6] The conclusions of this chapter do not mean that there is not another way to obtain the exact single–centered BH degeneracies after resummation of the residues of the RQDs in a manner consistent with the $Sp(2,\mathbb{Z})$ symmetry of Φ_{10}. We note, however, that such an enterprise would involve some notion of a "mock" Siegel form that has not been made precise in the mathematical literature to the best of the author's knowledge.

(b) Straight lines connecting i∞ to an integer. These can be thought of as special cases of the above expressions when $r = 0$ and $p = s = 1$, or when $s = 0$ and $q = -r = 1$. We denote these walls as *T–walls*.

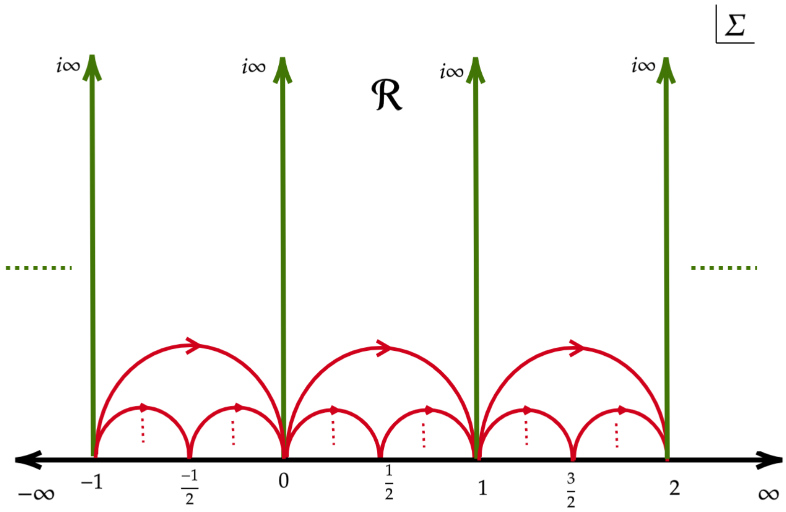


FIGURE 10. Structure of T–walls (green) and S–walls (red) in the upper half–plane.

To any T- or S–wall we associate the following matrix,

$$\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z})\,. \tag{8.23}$$

(2) Given an initial charge vector $(n,\ell,m) = (Q^2/2,\, Q\cdot P,\, P^2/2)$, there is an associated charge breakdown at a wall γ of the form (8.23), given by

$$\begin{pmatrix} Q \\ P \end{pmatrix} \longrightarrow \begin{pmatrix} p(sQ-qP) \\ r(sQ-qP) \end{pmatrix} + \begin{pmatrix} q(-rQ+pP) \\ s(-rQ+pP) \end{pmatrix}\,, \tag{8.24}$$

and which corresponds to a $\frac{1}{4}$–BPS BH decaying into two $\frac{1}{2}$–BPS centers. The charges of the two centers are given by $\gamma\cdot\begin{pmatrix} Q_\gamma \\ 0 \end{pmatrix}$ and $\gamma\cdot\begin{pmatrix} 0 \\ P_\gamma \end{pmatrix}$, with

$$\begin{pmatrix} Q_\gamma \\ P_\gamma \end{pmatrix} = \gamma^{-1}\cdot\begin{pmatrix} Q \\ P \end{pmatrix}\,, \tag{8.25}$$

which shows that after the breakdown one center is purely electric while the other is purely magnetic in the new frame. We define $(n_\gamma,\ell_\gamma,m_\gamma) = (Q_\gamma^2/2,\, Q_\gamma\cdot P_\gamma,\, P_\gamma^2/2)$, which are given explicitly by

$$\begin{aligned} n_\gamma &= s^2n + q^2m - sq\ell\,, \\ \ell_\gamma &= -2srn - 2pqm + \ell(ps+qr)\,, \\ m_\gamma &= r^2n + p^2m - pr\ell\,. \end{aligned} \tag{8.26}$$

(3) The set of matrices that characterize the walls in the Σ upper half–plane can be divided into subsets that satisfy the following properties:

$$\begin{aligned} \Gamma_S^+ &\coloneqq \left\{ \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z}) \;\Big|\; r > 0,\, s > 0 \right\}, \\ \Gamma_S^- &\coloneqq \left\{ \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z}) \;\Big|\; r > 0,\, s < 0 \right\}, \\ \Gamma_T &\coloneqq \left\{ \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z}) \;\Big|\; rs = 0 \right\}. \end{aligned} \tag{8.27}$$

Because the above matrices have unit determinant, the walls in Γ_S^+ have $p/r > q/s$, and the walls in Γ_S^- have $p/r < q/s$. We denote by $\Gamma_S = \Gamma_S^- \cup \Gamma_S^+$ the full set of S–walls. Notice that $PSL(2,\mathbb{Z}) = \Gamma_S \cup \Gamma_T$.

(4) We define the *orientation* of a wall γ to be $q/s \to p/r$. With respect to this orientation, a bound state of $\frac{1}{2}$–BPS states exists in the chamber to the right of the wall if $\ell_\gamma < 0$, and in the chamber to the left of the wall if $\ell_\gamma > 0$ [**Sen11b**].

(5) The attractor region $\mathcal{R}$ in (8.12) is the region of the Σ upper half–plane bounded by the T–walls $0 \to \mathrm{i}\infty$, $1 \to \mathrm{i}\infty$ and the semi–circular S–wall $0 \to 1$. (Note that, because of the negative sign in the contour (8.10), the attractor region $-\mathrm{Im}(\tau) < \mathrm{Im}(z) \leq 0$ as in (8.12) maps to $0 \leq \mathrm{Re}(\Sigma) < 1$.) We will be interested in the degeneracies of negative discriminant states in this region.

From Point 4 combined with (8.26), it is clear that none of the T–walls contribute in the region $\mathcal{R}$. For example, when $r = 0$, $\gamma = \left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right)$ and this implies $(n_\gamma, \ell_\gamma, m_\gamma) = (n + q^2 m - q\ell, \ell - 2qm, m)$. Recall that we can restrict ourselves to $0 \leq \ell < 2m$, this shows that when $q \geq 0$, the above T–walls contribute to the right of the region $\mathcal{R}$. This is consistent with the fact that ψ_m^{P}, which captures all the T–walls, actually has vanishing Fourier coefficients in the region $\mathcal{R}$.

For the S–walls, there exists a map between the sets Γ_S^+ and Γ_S^-, given by the right multiplication of an element of Γ_S^+ by the matrix

$$\tilde{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{8.28}$$

This map reverses the orientation of the wall and flips the sign of ℓ_γ. Furthermore, $\tilde{S}$ squares to $-I$, which means that it is an involution in $PSL(2,\mathbb{Z})$. Therefore we can focus only on elements of Γ_S^+ when discussing the details of negative discriminant states breakdowns across walls of marginal stability.

8.4.2. Towards a formula for black hole degeneracies. Upon crossing a wall of marginal stability, the index jumps by an amount controlled by the generating function of each of the associated $\frac{1}{2}$–BPS centers. The latter is given by the inverse of $\eta(\tau)^{24}$, whose Fourier coefficients are given by the partition function into 24 colors $p_{24}(n)$ (cf. Equation (8.6)). Summing up all possible decays across the S–walls leads to the following counting formula for negative discriminant states living in the region $\mathcal{R}$:

$$\frac{1}{2} \sum_{\gamma \in \Gamma_S} (-1)^{\ell_\gamma + 1}\, \theta(\gamma, \mathcal{R})\, |\ell_\gamma|\, d(m_\gamma)\, d(n_\gamma)\,, \tag{8.29}$$

where the function $\theta(\gamma,\mathcal{R})$ is a step–function giving 1 if the bound state exists on the same side of the wall γ bounding $\mathcal{R}$ and 0 otherwise. Formally, it is defined as follows

$$\theta(\gamma,\mathcal{R}) \;=\; \left|\frac{\mathcal{O}(\gamma,\mathcal{R})+\mathrm{sgn}(\ell_\gamma)}{2}\right|, \qquad \mathcal{O}(\gamma,\mathcal{R}) \;=\; \begin{cases} +1, & \gamma\in\Gamma_S^+ \\ -1, & \gamma\in\Gamma_S^- \end{cases}\,. \tag{8.30}$$

On one hand this sum can be written in a more covariant manner by extending it to a sum over all matrices in $PSL(2,\mathbb{Z})$,

$$\frac{1}{2}\sum_{\gamma\in PSL(2,\mathbb{Z})}(-1)^{\ell_\gamma+1}\,\theta(\gamma,\mathcal{R})\,|\ell_\gamma|\,d(m_\gamma)\,d(n_\gamma)\,, \tag{8.31}$$

by extending the θ function to all of $PSL(2,\mathbb{Z})$ via

$$\theta(\gamma,\mathcal{R}) \;=\; 0\,, \qquad \gamma\in\Gamma_T\,, \tag{8.32}$$

because the T–walls do not contribute in the region $\mathcal{R}$ as we saw above. On the other hand, the sum (8.30) can also be written as a sum over a smaller set as follows. Note that the summand in equation (8.29) is invariant under a transformation by the matrix $\tilde S$ given in equation (8.28) because $n_{\gamma\tilde S}=m_\gamma$, $m_{\gamma\tilde S}=n_\gamma$, $\ell_{\gamma\tilde S}=-\ell_\gamma$ and $\tilde S$ exchanges Γ_S^+ and Γ_S^-. This means that the contributions from the sum over $\gamma\in\Gamma_S^+$ and $\gamma\in\Gamma_S^-$ are equal and we can sum over Γ_S^+ only. So, we can alternatively write (8.29) as

$$\sum_{\gamma\in\Gamma_S^+}(-1)^{\ell_\gamma+1}\left|\frac{1+\mathrm{sgn}(\ell_\gamma)}{2}\right|\,|\ell_\gamma|\;d(m_\gamma)\;d(n_\gamma)\,. \tag{8.33}$$

8.4.3. A subtlety from bound state metamorphosis. While accounting for all the negative discriminant states in the region $\mathcal{R}$, there is a further subtlety that needs to be taken into account due to a phenomenon known as bound state metamorphosis (BSM) [**ADJM12, CLSS13, Sen11b**]. BSM stems from the fact that when one or both $\frac{1}{2}$–BPS centers making up a $\frac{1}{4}$–BPS bound state carry the lowest possible charge invariant (that is, when $n_\gamma=-1$, $m_\gamma=-1$ or $n_\gamma=m_\gamma=-1$ for a given wall γ), two or more bound states must be identified following a precise set of rules to avoid overcounting in the index (8.33). Thus we can write the set of all contributing walls as the quotient

$$\Gamma_{\mathrm{BSM}}(n,\ell,m) \;=\; PSL(2,\mathbb{Z})/\mathrm{BSM}\,, \tag{8.34}$$

and write the polar degeneracies (8.8) as

$$\widetilde{c}_m(n,\ell) \;=\; \frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{BSM}}(n,\ell,m)}(-1)^{\ell_\gamma+1}\,\Theta(\gamma)\,|\ell_\gamma|\,d(m_\gamma)\,d(n_\gamma)\,. \tag{8.35}$$

Here we have to introduce a new function $\Theta(\gamma)$ which generalizes the function $\theta(\gamma,\mathcal{R})$ defined above to take into account the phenomenon of BSM so that it is defined on the coset $\Gamma_{\mathrm{BSM}}(n,\ell,m)$. We will be in a position to give a proper definition after a discussion of BSM in the following sections. We can also present this formula as a sum over the set Γ_S or Γ_S^+ modulo the identifications due to BSM for the reasons discussed above (T–walls do not contribute in $\mathcal{R}$, and $\tilde S$ gives a map between Γ_S^- and Γ_S^+):

$$\widetilde{c}_m(n,\ell) \;=\; \sum_{\gamma\in\Gamma_S^+/\mathrm{BSM}}(-1)^{\ell_\gamma+1}\,\Theta(\gamma)\,|\ell_\gamma|\;d(m_\gamma)\;d(n_\gamma)\,. \tag{8.36}$$

Given that the left–hand side of this formula is finite, it is reasonable to expect that given a set of initial charges (n,ℓ,m), only a finite subset of walls of marginal stability gives a non–zero contribution to the above sums. This expectation turns out to be correct and we can write the final formula as a sum over the *finite* set $\mathrm{W}(n,\ell,m)$

$$\widetilde{c}_m(n,\ell) \;=\; \sum_{\gamma\in\mathrm{W}(n,\ell,m)} (-1)^{\ell_\gamma+1}\,|\ell_\gamma|\; d(m_\gamma)\; d(n_\gamma)\,. \tag{8.37}$$

The goal in the following sections is to now fully characterize the subset $\mathrm{W}(n,\ell,m)$ and show that it contains a finite number of elements for a given charge vector (n,ℓ,m).

It will be convenient to split the characterization of the set $\mathrm{W}(n,\ell,m)$ depending on whether BSM does not or does occur. This will be the subject of sections 8.5 and 8.6, respectively. Before initiating the study of the finiteness of $\mathrm{W}(n,\ell,m)$, we recall from the discussion below (8.11) that we can restrict the charge vector to be such that $0\le\ell<2m$. In addition, ψ^F_m has even weight so there is a reflection symmetry $\ell\to-\ell$ which allows us to restrict ourselves to the case[7] $\ell\in\{0,\dots,m\}$. The index m runs from -1 to $+\infty$ in the expansion (7.17). For $m=-1$ and $m=0$, there is no macroscopic BH and therefore we only study $m>0$ in the following. Thus the goal is to study the set $\mathrm{W}(n,\ell,m)$ of walls of marginal stability for a charge vector (n,ℓ,m) that satisfies $m>0$, $n\ge-1$ and $0\le\ell\le m$.

8.5. Negative discriminant states without metamorphosis

In this section, we begin to characterize $\mathrm{W}(n,\ell,m)$. For the time being we ignore the phenomenon of BSM (which will be the subject of the next section), and show that the contribution to $\mathrm{W}(n,\ell,m)$ in this case is finite. Accordingly, in order to identify the walls that contribute to the polar degeneracies $\widetilde{c}_m(n,\ell)$, we study the system of inequalities $m_\gamma\ge0$ and $n_\gamma\ge0$ defined in (8.26) for a given charge vector (n,ℓ,m) such that $\Delta=4mn-\ell^2<0$ and $0\le\ell\le m$. As explained above, we focus on walls in $\Gamma^+_S\subset PSL(2,\mathbb{Z})$, which allows us to choose $r,s>0$ in the following. The condition $m_\gamma\ge0$ then amounts to

$$m_\gamma = m\left(\frac{p}{r}\right)^2 - \ell\left(\frac{p}{r}\right) + n \ge 0\,. \tag{8.38}$$

The first equality defines a parabola in the $(p/r, y=m_\gamma)$–plane and the condition $m>0$ means that the inequality has two branches:

$$\frac{p}{r}\ge\frac{\ell+\sqrt{|\Delta|}}{2m}\quad\text{or}\quad\frac{p}{r}\le\frac{\ell-\sqrt{|\Delta|}}{2m}\,. \tag{8.39}$$

We will call these positive and negative "runaway branches" since p/r is unbounded from above or from below, respectively. The condition $n_\gamma\ge0$ amounts to

$$n_\gamma = m\left(\frac{q}{s}\right)^2 - \ell\left(\frac{q}{s}\right) + n \ge 0\,. \tag{8.40}$$

Moreover, using that the determinant of γ must be equal to one, we have

$$\frac{q}{s}=\frac{p}{r}-\frac{1}{rs}\,, \tag{8.41}$$

[7] Note that even though ψ^F_m is not modular but only mock modular, both its completion $\widehat{\psi}^F_m$ and its shadow, and therefore ψ^F_m itself, enjoy this $\ell\to-\ell$ symmetry [**DMZ12**].

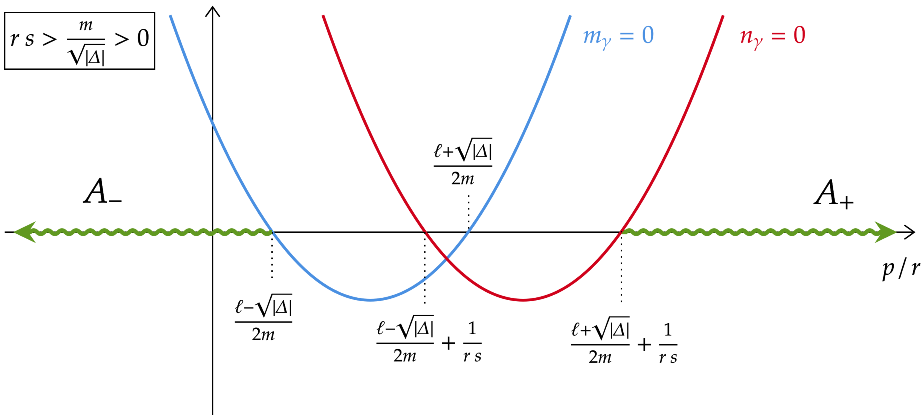
and so the first equality in (8.40) can also be seen as a parabola in the $(p/r, y = n_\gamma)$−plane, shifted by $1/(rs)$ compared to the first parabola. The condition $n_\gamma \geq 0$ also has a positive and negative runaway branch,

$$\frac{p}{r} \geq \frac{\ell + \sqrt{|\Delta|}}{2m} + \frac{1}{rs} \quad \text{or} \quad \frac{p}{r} \leq \frac{\ell - \sqrt{|\Delta|}}{2m} + \frac{1}{rs}\,. \tag{8.42}$$

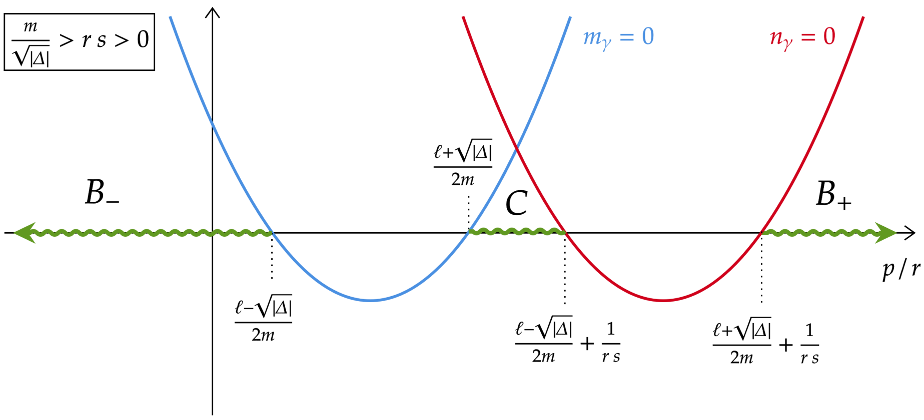
Recall from Section 8.4 that we focus on Γ_S^+ walls which corresponds to $rs > 0$. We split the argument in two cases. Considering

$$\frac{\ell - \sqrt{|\Delta|}}{2m} + \frac{1}{rs} < \frac{\ell + \sqrt{|\Delta|}}{2m}\,, \tag{8.43}$$

the smaller intercept with the p/r axis of the shifted n_γ-parabola is smaller than the larger intercept with the p/r axis of the m_γ-parabola. This situation is illustrated in Figure 11a. In this case, requiring both inequalities $m_\gamma \geq 0$ and $n_\gamma \geq 0$ implies that



(A) The two runaway branches $A_\pm$ in the case where $rs > \frac{m}{\sqrt{\Delta}} > 0$



(B) The runaway branches $B_\pm$ and the bounded branch C when $\frac{m}{\sqrt{\Delta}} > rs > 0$

FIGURE 11. The regions where $m_\gamma \geq 0$ and $n_\gamma \geq 0$ for $rs > 0$, denoted in green

$$\frac{p}{r} \geq \frac{\ell + \sqrt{|\Delta|}}{2m} + \frac{1}{rs} \quad \text{and} \quad rs > \frac{m}{\sqrt{|\Delta|}} > 0\,, \tag{8.44}$$

on the positive runaway branch which we denote A_+, or

$$\frac{p}{r} \leq \frac{\ell - \sqrt{|\Delta|}}{2m} \quad \text{and} \quad rs > \frac{m}{\sqrt{|\Delta|}} > 0\,, \tag{8.45}$$

on the negative runaway branch which we denote A_-. For a given set of (n,ℓ,m), the conditions (8.44) have solutions over the integers for (p,r,s). However, if we supplement this system with the condition that $\ell_\gamma > 0$ (since this is a Γ_S^+ wall this condition is necessary to have a non–zero contribution owing to the Θ function in (8.36)), then there are no solutions for (p,r,s). Indeed these conditions imply that

$$\begin{aligned} &0 < 2\,\frac{r}{s}\,n_\gamma + \ell_\gamma = \ell - 2\,\frac{q}{s}\,m = \ell - 2\left(\frac{p}{r} - \frac{1}{rs}\right)m \\ \Longrightarrow\quad &\frac{p}{r} < \frac{\ell}{2m} + \frac{1}{rs}\,, \end{aligned} \tag{8.46}$$

which is in contradiction with the first equality of (8.44). Similarly, supplementing the branch A_- by the condition $\ell_\gamma > 0$, there are no solutions for (p,r,s). This can be seen by showing that the inequality $2\frac{s}{r}m_\gamma + \ell_\gamma > 0$ is in contradiction with the first inequality of (8.45).

The other case to consider is when

$$\frac{\ell - \sqrt{|\Delta|}}{2m} + \frac{1}{rs} \geq \frac{\ell + \sqrt{|\Delta|}}{2m}\,. \tag{8.47}$$

This means that the smaller intercept of the shifted parabola is larger than or equal to the larger intercept of the original one, as illustrated in Figure 11b. In this case, we still have the usual runaway branches which we call B_+ and B_-, but in addition a new branch of solutions for p/r opens up, which we call the bounded branch C,

$$\frac{\ell + \sqrt{|\Delta|}}{2m} \leq \frac{p}{r} \leq \frac{\ell - \sqrt{|\Delta|}}{2m} + \frac{1}{rs} \quad \text{and} \quad \frac{m}{\sqrt{|\Delta|}} \geq rs > 0\,. \tag{8.48}$$

Once again, adding the condition that $\ell_\gamma > 0$ suffices to show that there are no integer solutions (p,r,s) to the system of inequalities characterizing the runaway branches $B_\pm$. The proof of this is identical to the one above for $A_\pm$. There are now however solutions for the C branch. Observe that on this branch we also have a condition on the original charges: since r and s are integers, $rs \geq 1$ and so the second condition in (8.48) demands that $m \geq \sqrt{|\Delta|}$. Including the inequality $\ell_\gamma > 0$, we obtain the following system for potential walls without BSM contributing to the polar coefficients:

$$\left\{ \begin{array}{l} \frac{\ell+\sqrt{|\Delta|}}{2m} \leq \frac{p}{r} \leq \frac{\ell-\sqrt{|\Delta|}}{2m} + \frac{1}{rs} \\ \frac{m}{\sqrt{|\Delta|}} \geq rs > 0 \\ -2nrs - 2mpq + \ell(ps+qr) > 0 \end{array} \right. . \tag{8.49}$$

To analyze this system, we start by using the unit determinant condition to eliminate q, and then express everything in terms of the variables

$$P := ps\,, \quad R := rs\,. \tag{8.50}$$

Then (8.49) takes the form of a system of inequalities on two variables (P, R):

$$\begin{cases} \frac{\ell+\sqrt{|\Delta|}}{2m} \leq \frac{P}{R} \leq \frac{\ell-\sqrt{|\Delta|}}{2m} + \frac{1}{R} \\ \frac{m}{\sqrt{|\Delta|}} \geq R > 0 \\ -2nR - 2m\frac{P}{R}(P-1) + \ell(2P-1) > 0 \end{cases} . \tag{8.51}$$

We can analyze this system on a case-by-case basis, depending on the original charges (n, ℓ, m). Recall that we must only consider $4mn - \ell^2 < 0$, $m > 0$ and $0 \leq \ell \leq m$.

(1) **Case 1**: $m > 0$, $n = -1$
In this case, there are no integer solutions to (8.51). We see from equations (8.26) that $n = -1$ and $n_\gamma \geq 0$, $m_\gamma \geq 0$ require that $p, q \neq 0$. The left-hand-side of the first inequality in equation (8.51) implies that $P/R > 0$, which then implies that $p > 0$ since $r, s > 0$. The determinant condition $ps - qr = 1$ then requires that $q > 0$ as well. However, this is a contradiction since the right-hand-side of the first inequality in equation (8.51) can be rewritten as

$$\frac{q}{s} = \frac{P}{R} - \frac{1}{R} \leq \frac{\ell - \sqrt{|\Delta|}}{2m} < 0\,. \tag{8.52}$$

Here we used that for $m > 0, n = -1$ we have $\sqrt{|\Delta|} = \sqrt{\ell^2 + 4m} > \ell$.

(2) **Case 2**: $m > 0$, $n = 0$, and $\ell > 0$
In this case we find solutions given by

$$P = 1\,, \quad 0 < R \leq \frac{m}{\ell}\,. \tag{8.53}$$

Translating back to the original (p, q, r, s) variables, this yields matrices of the form

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}\,, \quad \text{with} \quad 0 < r \leq \frac{m}{\ell}\,. \tag{8.54}$$

Note that all entries in the above matrix are bounded from above by m. As we will see below, such m-dependent bounds always arise when considering the set of contributing walls $\mathrm{W}(n, \ell, m)$.

(3) **Case 3**: $m > 0$, $n > 0$ and $\ell > 0$
This case is slightly more involved. First, notice from the left-hand-side of the first inequality in equation (8.51) that $P = ps > 0$. Therefore we split the discussion depending on whether $P = 1$ or $P > 1$.

(a) **Case 3a:** $P = 1$
In this case the inequalities (8.51) with $P = 1$ impose

$$P = 1\,, \quad 0 < R \leq \frac{\ell - \sqrt{|\Delta|}}{2n}\,. \tag{8.55}$$

In the variables (p, q, r, s), we therefore have a non–zero contribution to the polar coefficients from matrices of the form:

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}\,, \quad \text{with} \quad 0 < r \leq \frac{\ell - \sqrt{|\Delta|}}{2n} \leq \frac{m-1}{2n}\,. \tag{8.56}$$

Again, note that all entries in the above matrix are smaller than m.

(b) **Case 3b:** $P > 1$
In this case, the inequalities (8.51) yield the following bounds on P and R:

$$1 < P \leq \frac{1}{2}\Big(1 + \frac{\ell}{\sqrt{|\Delta|}}\Big)\,, \qquad \frac{\ell + \sqrt{|\Delta|}}{2n}\,(P-1) \leq R \leq \frac{\ell - \sqrt{|\Delta|}}{2n}\,P\,. \tag{8.57}$$

The corresponding walls do not start at $q/s = 0$ but instead are strictly inside the largest semi-circular S-wall $0 \to 1$.
Note that we can again get an m-dependent upper bound on P by using the fact that $\ell \leq m$ and $\sqrt{|\Delta|} \geq 1$. We can also use this upper bound on P in the upper bound on R directly to obtain

$$P \leq \frac{m+1}{2}\,, \qquad R \leq \frac{m}{\sqrt{|\Delta|}}\,. \tag{8.58}$$

The above implies that all matrix entries of $\gamma \in PSL(2,\mathbb{Z})$ satisfying (8.57) are bounded from above by m.

This exhausts all possible cases for contributions without BSM: the conditions (8.54), (8.56) and (8.57) with $n \geq 0$ and $\ell, m > 0$ fully characterize the set $\mathrm{W}(n,\ell,m)$ in this case. By inspection, this set has a finite number of elements. Observe that all the walls giving a non-zero contribution to (8.37) have entries bounded from above by m.

8.6. Effects of black hole bound state metamorphosis

We now turn to identifying the walls of marginal stability for which BSM is relevant. We study the problem systematically in three different cases viz., magnetic-, electric-, and dyonic-metamorphosis, corresponding to $m_\gamma = -1$, $n_\gamma = -1$ and $m_\gamma = n_\gamma = -1$, respectively.

8.6.1. Magnetic metamorphosis case: $m_\gamma = -1, n_\gamma \geq 0$. As in Section 8.5, we start with a charge vector (n,ℓ,m) such that $4mn - \ell^2 < 0$ and $0 \leq \ell \leq m$. However, we are now interested in the walls γ for which $m_\gamma = -1$ and $n_\gamma \geq 0$. The idea behind magnetic metamorphosis is that when there is a wall γ such that $m_\gamma = -1$, then there is another wall $\tilde\gamma$ which has the exact same contribution to the index as γ. Furthermore, one needs to implement a precise prescription to properly account for such walls, and avoid overcounting in the polar degeneracies, as shown in [**Sen11b, CLSS13**]. It will be useful to explicitly review some details of this phenomenon. To do so, we begin with the following definition:

DEFINITION 8.6.1. *For a wall γ with $m_\gamma = -1, n_\gamma \geq 0$, we define its metamorphic dual as*

$$\tilde\gamma \;:=\; \gamma \cdot \begin{pmatrix} 1 & -\ell_\gamma \\ 0 & 1 \end{pmatrix}\,. \tag{8.59}$$

With this definition, the prescription found in [**Sen11b, CLSS13**] to properly account for magnetic-BSM can be summarized as follows (we refer the reader to the references just mentioned for a physical justification of this):

A wall γ at which magnetic-BSM occurs contributes to the polar coefficients $\widetilde{c}_m(n,\ell)$ if and only if γ and its metamorphic dual $\tilde{\gamma}$ both contribute in $\mathcal{R}$. If so, the contributions of γ and $\tilde{\gamma}$ should be counted only once.

A necessary condition for the second part of the prescription is that both γ and $\tilde{\gamma}$ have the same index contribution, as we now review. First, from Definition 8.6.1, it is easy to see that a wall γ and its metamorphic dual $\tilde{\gamma}$ have the same end point p/r,

$$\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \iff \tilde{\gamma} = \begin{pmatrix} p & -p\ell_\gamma + q \\ r & -r\ell_\gamma + s \end{pmatrix} . \tag{8.60}$$

With this, we prove the following statement:

Proposition 8.6.1. *For a given set of charges (n,ℓ,m), the wall $\tilde{\gamma}$ has the same index contribution as γ to the polar coefficients $\widetilde{c}_m(n,\ell)$.*

Proof. First consider the wall γ for which the electric and magnetic centers are

$$Q_\gamma = sQ - qP\,, \quad P_\gamma = -rQ + pP\,. \tag{8.61}$$

Thus, we have

$$\ell_\gamma = Q_\gamma \cdot P_\gamma = -srQ^2 - qpP^2 + (sp+qr)\,Q\cdot P. \tag{8.62}$$

A similar calculation for $\tilde{\gamma}$ shows that $Q_{\tilde{\gamma}} = Q_\gamma + \ell_\gamma P_\gamma$ and $P_{\tilde{\gamma}} = P_\gamma$, as well as

$$\ell_{\tilde{\gamma}} = (Q_\gamma + \ell_\gamma P_\gamma)\cdot P_\gamma = \ell_\gamma + \ell_\gamma P_\gamma^2 = -\ell_\gamma\,, \tag{8.63}$$

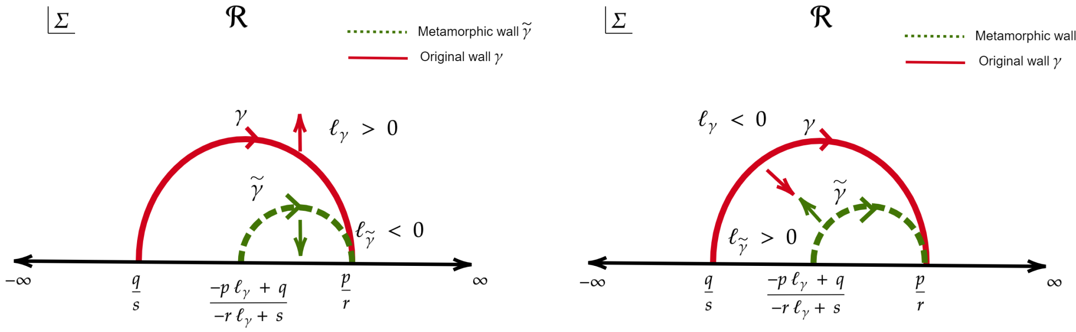
where in the last equality we have made use of the fact that $m_\gamma = P_\gamma^2/2 = -1$. We also note that the above considerations imply

$$m_\gamma = m_{\tilde{\gamma}} = -1\,, \quad n_{\tilde{\gamma}} = \left(n_\gamma + \ell_\gamma^2 m_\gamma + \ell_\gamma^2\right) = n_\gamma\,. \tag{8.64}$$

From (8.63), (8.64) and the fact that a given bound state of charges (n',ℓ',m') has indexed degeneracy $(-1)^{\ell'+1}|\ell'|d(n')d(m')$, we conclude that a wall γ and its metamorphic image $\tilde{\gamma}$ contribute equally to the negative discriminant degeneracies. □

We now illustrate the potential non-zero contributions to the formula (8.36) from magnetic-BSM walls, implementing the above prescription. Recall that the walls we are summing over in the index formula are in Γ_S^+ and are thus oriented from left to right (they have $p/r > q/s$). From (8.60), we then have the following possible configurations for the wall γ and its dual:

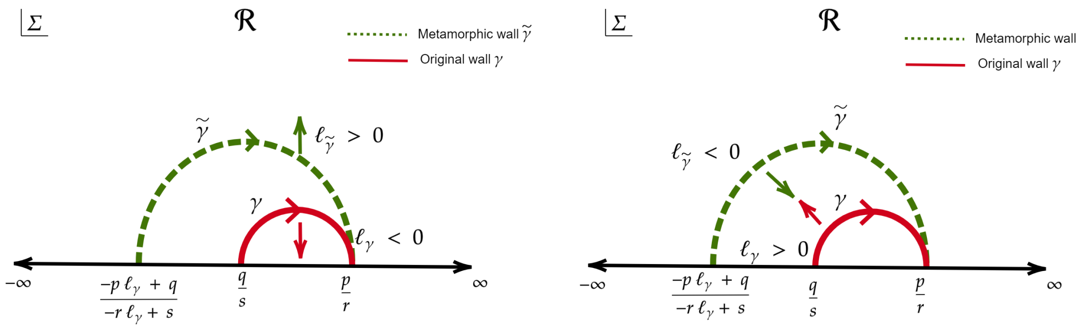
(1) **Case** $\frac{p}{r} > \frac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \frac{q}{s}$, as shown in Figure 12
 (a) The situation $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$ shown in Figure 12a leads to a contradiction as follows. If $-r\ell_\gamma + s > 0$, then we find from $\frac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \frac{q}{s}$ that $-ps\ell_\gamma + qs > -rq\ell_\gamma + qs$ which implies $0 > \ell_\gamma(ps - qr) = \ell_\gamma$ (because $\gamma \in PSL(2,\mathbb{Z})$), which contradicts the assumption that $\ell_\gamma > 0$. For $-r\ell_\gamma + s < 0$, we find from $\frac{p}{r} > \frac{-p\ell_\gamma + q}{-r\ell_\gamma + s}$ that $-pr\ell_\gamma + ps < -pr\ell_\gamma + qr$ which leads to the contradiction $1 = ps - qr < 0$. Therefore this scenario does not occur.
 (b) The situation $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$ as seen in Figure 12b does occur but does not contribute to the index computed in the region $\mathcal{R}$ owing to the BSM prescription presented below Definition 8.6.1.



(A) Case $\dfrac{p}{r} > \dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \dfrac{q}{s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$ (B) Case $\dfrac{p}{r} > \dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \dfrac{q}{s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$

FIGURE 12. Metamorphosis for $\dfrac{p}{r} > \dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \dfrac{q}{s}$

(2) **Case** $\dfrac{p}{r} > \dfrac{q}{s} > \dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s}$, as shown in Figure 13

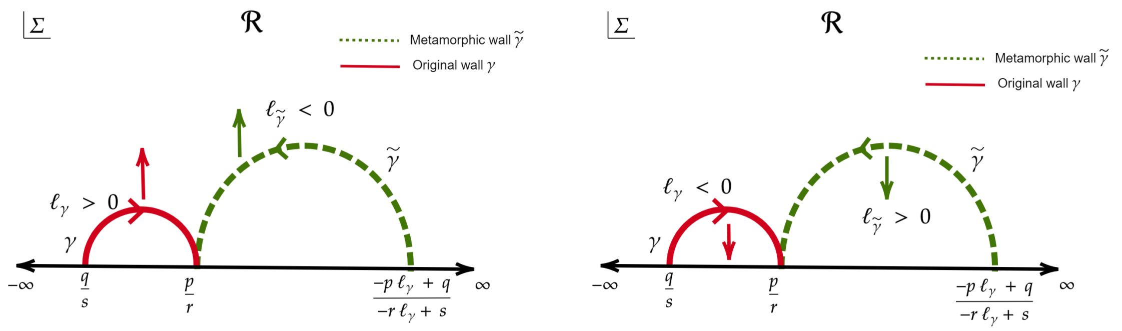


(A) Case $\dfrac{p}{r} > \dfrac{q}{s} > \dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$ (B) Case $\dfrac{p}{r} > \dfrac{q}{s} > \dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$

FIGURE 13. Metamorphosis for $\dfrac{p}{r} > \dfrac{q}{s} > \dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s}$

(a) For the case $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$ as in Figure 13a, we run into a contradiction analogous to the one of Figure 12a.
(b) The case of $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$ as in Figure 13b does again occur but does not contribute to the black hole degeneracy in the region $\mathcal{R}$ owing to the BSM prescription.

(3) **Case** $\dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \dfrac{p}{r} > \dfrac{q}{s}$ as shown in Figure 14
(a) For the case as in Figure 14a, there will be a contribution to the index in the region $\mathcal{R}$ from the walls γ and $\tilde{\gamma}$, in accordance with the BSM prescription. Furthermore, Proposition 8.6.1 shows that both contributions are equal, and the prescription states that they must be identified to avoid overcounting.
(b) The case shown in Figure 14b does not contribute to the black hole degeneracy in $\mathcal{R}$ since neither γ nor $\tilde{\gamma}$ contribute in $\mathcal{R}$.



(A) Case $\dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \dfrac{p}{r} > \dfrac{q}{s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$ (B) Case $\dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \dfrac{p}{r} > \dfrac{q}{s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$

FIGURE 14. Metamorphosis for $\dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \dfrac{p}{r} > \dfrac{q}{s}$

In summary, we have shown that the only magnetic-BSM walls that give a non-trivial contribution to (8.36) must satisfy $m_\gamma = -1$, $n_\gamma \geq 0$, $\ell_\gamma > 0$, as well as $\dfrac{-p\ell_\gamma + q}{-r\ell_\gamma + s} > \dfrac{p}{r}$. Observe now that if $-r\ell_\gamma + s > 0$, then we can rewrite the latter inequality as $0 > ps - qr = 1$ which is a contradiction. Thus, we find that the walls giving a non-trivial contribution to the index (8.36) must have $-r\ell_\gamma + s < 0$, which implies $\ell_\gamma > s/r$ and therefore is stronger than $\ell_\gamma > 0$.

Upon eliminating q using the condition that the walls are in $PSL(2,\mathbb{Z})$, we can write the three conditions $m_\gamma = -1$, $n_\gamma \geq 0$ and $\ell_\gamma > s/r$ as

$$
\begin{aligned}
m\Big(\frac{ps-1}{r}\Big)^2 - \ell\Big(\frac{ps-1}{r}\Big)s + ns^2 &\geq 0\,, \\
mp^2 - \ell pr + nr^2 &= -1\,, \\
-2nrs - 2m\frac{p}{r}(ps-1) + \ell(2ps-1) &> \frac{s}{r}\,.
\end{aligned}
\tag{8.65}
$$

We split the discussion in various cases depending on the values of the charges (n, ℓ, m), subject to the conditions $4mn - \ell^2 < 0$, $m > 0$ and $0 \leq \ell \leq m$. We further focus on the Γ_S^+ walls that have $r, s > 0$.

(1) **<u>Case 1</u>**: $m > 0$, $n = -1$
In this case we solve for r in (8.65) and obtain two solutions

$$
r_\pm = \frac{1}{2}\Big(\pm\sqrt{p^2|\Delta| + 4} - \ell p\Big)\,. \tag{8.66}
$$

Since r_- is negative we can discard it and focus on the r_+ solution. Inserting this in the inequalities $n_\gamma \geq 0$ and $\ell_\gamma > s/r$, we obtain the following inequalities on s,

$$
\max\Big[\frac{1}{2}\Big(\ell\sqrt{p^2|\Delta| + 4} - p|\Delta|\Big), 0\Big] < s \leq \frac{1}{4}\Big(\ell + \sqrt{|\Delta|}\Big)\Big(\sqrt{p^2|\Delta| + 4} - p\sqrt{|\Delta|}\Big)\,, \tag{8.67}
$$

where we have taken into account the fact that we are only interested in solutions with $s > 0$. Clearly the right-hand side must be greater or equal to one for this to have solutions in $\mathbb{Z}$, which in turn translates to an upper bound on p, given below. A lower bound on p arises because the lower bound on s will become p-dependent for sufficiently small p (certainly for $p \leq 0$). In that case we know that the lower bound $\frac{1}{2}\Big(\ell\sqrt{p^2|\Delta|+4}-p|\Delta|\Big) = \frac{1}{2}\Big(\ell(2r_+ + \ell p)-p|\Delta|\Big)$ is an integer or a half-integer. Since s has to be strictly larger than this lower bound but smaller than the upper bound, we find that the gap between the upper and lower bound on s has to be at least $1/2$, which leads to a lower bound on p. The final range one obtains is

$$-1+\frac{1}{4m}\left(\frac{\ell(1+4m)}{\sqrt{|\Delta|}+1}\right) \leq p \leq \frac{1}{2}+\frac{1}{2m}\left(\frac{\ell(m+1)}{\sqrt{|\Delta|}}-1\right). \tag{8.68}$$

Since $0 \leq \ell \leq m$, the upper bound on p is maximized by taking $\ell = m$, in which case one obtains $p \leq \frac{1}{2}\Big(1-\frac{1}{m}+\frac{m+1}{\sqrt{m(m+4)}}\Big) < 1$, while the lower bound on p trivially implies that $p \geq 0$. So, we actually find that there are only solutions with $p = 0$, which then implies that $r = r_+ = 1$ and $q = (ps-1)/r = -1$. The range for s simplifies substantially and the only matrices that contribute in this case are

$$\begin{pmatrix} 0 & -1 \\ 1 & s \end{pmatrix}, \quad \text{with} \quad \ell < s \leq \frac{1}{2}\left(\ell+\sqrt{|\Delta|}\right) < m+1\,. \tag{8.69}$$

Here we have used that $0 \leq \ell \leq m$ to get a simple m-dependent upper bound on s.

(2) **Case 2**: $m > 0$, $n = 0$ and $\ell > 0$
In this case, the system (8.65) imposes

$$r = \frac{1+mp^2}{\ell p}\,. \tag{8.70}$$

Requiring $r > 0$ to be an integer fixes $p = 1$ and $\ell\,|\,(m+1)$. Then the condition $\ell_\gamma > s/r$ is automatically satisfied for $s > 0$, while the condition $m_\gamma \geq 0$ requires $s = 1$. Thus, the matrices satisfying (8.65) are of the form

$$\begin{pmatrix} 1 & 0 \\ \frac{m+1}{\ell} & 1 \end{pmatrix}, \quad \text{with} \quad \ell\,|\,(m+1)\,. \tag{8.71}$$

This set of matrices has entries that are trivially bounded from above by $(m+1)/\ell$. Among all matrices that contribute to the index (8.36), we obtain here the maximal entry $m+1$ for $\ell = 1$.

(3) **Case 3**: $m > 0$, $n > 0$ and $\ell > 0$
In this case we solve for r using $m_\gamma = -1$ and obtain two solutions

$$r_\pm = \frac{1}{2n}\left(\ell p \pm \sqrt{p^2|\Delta|-4n}\right). \tag{8.72}$$

Note that $\mathrm{sign}(p) = \mathrm{sign}(r_\pm)$, so our restriction to $r > 0$ implies in this case $p > 0$. The reality of the square root in $r_\pm$ actually implies a stronger lower bound on p,

$$2\sqrt{\frac{n}{|\Delta|}} \leq p\,. \tag{8.73}$$

Together with the conditions $n_\gamma \geq 0$ and $\ell_\gamma > s/r$, we find

(8.74)
$$\max\left[\frac{1}{2n}\left(p|\Delta| \pm \ell\sqrt{p^2|\Delta|-4n}\right),0\right] < s \leq \frac{1}{4n}\left(\ell+\sqrt{|\Delta|}\right)\left(p\sqrt{|\Delta|} \pm \sqrt{p^2|\Delta|-4n}\right),$$

where the upper and lower signs are for $r = r_+$ and $r = r_-$, respectively. We should require that the right-hand side of the above equation be greater than one to have integer solutions for s. For the lower sign, this criterion yields an upper bound on p,

(8.75)
$$p \leq \frac{1}{2} + \frac{1}{2m}\left(\frac{\ell(m+1)}{\sqrt{|\Delta|}} - 1\right),$$

which shows that there is a finite number of walls with $r = r_-$ contributing to (8.36). Furthermore, the wall matrix entries are again bounded by simple m-dependent functions, as follows. For p as in (8.75) we notice that the upper bound is maximized for $\ell = m$ and $|\Delta| = 1$,[8] in which case one finds $p < \frac{1}{2}\left(2+m-\frac{1}{m}\right) < 1+\frac{m}{2}$. Similarly, we can derive an m-dependent upper bound on $s > 0$ as follows: $p\sqrt{|\Delta|} - \sqrt{p^2|\Delta|-4n}$ is a monotonically decreasing function of p and therefore maximal when p is at its lower bound $2\sqrt{n/|\Delta|}$ from (8.73). Taking into account that $1 \leq n$ and $\sqrt{|\Delta|} < \ell \leq m$, we then find $s \leq \frac{\ell+\sqrt{|\Delta|}}{2\sqrt{n}} < m$. Using the upper bound on s we likewise find an upper bound on the remaining entry, $0 \leq q = (ps-1)/r_- \leq \left(p\sqrt{|\Delta|} - \sqrt{p^2|\Delta|-4n}\right)/2 < \sqrt{m}/2$.

For the upper sign, which corresponds to picking $r = r_+$ in (8.72), requiring that the right-hand side of (8.74) be greater than one does not yield additional constraints on p. In this case, we thus only have the lower bound (8.73). However, numerical investigations up to $m = 30$ show that the set of walls with $r = r_+$ is finite, and in fact consists of only a single element for a given value of $m, n, \ell > 0$. Furthermore, the entries of the matrix associated to such walls are always strictly less than m. It seems that imposing integrality of the matrix entries on top of the above conditions severely restricts the contributing walls with $r = r_+$, although we have not managed to show this analytically. We leave this as an interesting problem for the future.

The above analysis shows that the set of walls at which magnetic-BSM occurs and that give a non-trivial contribution to the index (8.36) is finite with entries bounded from above by $m+1$ (see Equation (8.71)). Aside from the case with $m, n, \ell > 0$ and $r = r_+$, we were able to show this analytically. Nevertheless, the numerical investigations have shown that the same conclusion holds for the latter walls. Some more details are presented in Appendix C.

8.6.2. Electric metamorphosis case: $n_\gamma = -1, m_\gamma \geq 0$. Having expounded the details of the magnetic metamorphosis case in the previous subsection, we can make use of these results to work out the electric metamorphosis case at almost no extra cost. This follows from combining Proposition 8.6.1 with the observation below (8.28), which shows that acting with $\tilde{S}$ on the metamorphic dual $\tilde{\gamma}$ (8.59) of a wall γ with $m_\gamma = -1$

[8] These values cannot actually be obtained, so there is a slightly stronger but more complicated bound.

produces a wall with $n_\gamma = -1$ and with the same orientation as that of γ. Indeed, the charges associated with the wall $\tilde{\gamma} \cdot \tilde{S}$ are given by

$$(n_{\tilde{\gamma}\tilde{S}},\, \ell_{\tilde{\gamma}\tilde{S}},\, m_{\tilde{\gamma}\tilde{S}}) \;=\; (m_\gamma,\, \ell_\gamma,\, n_\gamma) \;=\; (-1,\, \ell_\gamma,\, n_\gamma)\,. \tag{8.76}$$

As in the magnetic-BSM case, for a wall γ such that electric metamorphosis occurs there is another wall $\tilde{\gamma}$ which gives the same contribution to the index:

Definition 8.6.2. *For a wall γ with $n_\gamma = -1,\; m_\gamma \geq 0$, we define its metamorphic dual as*

$$\tilde{\gamma} = \gamma \cdot \begin{pmatrix} 1 & 0 \\ -\ell_\gamma & 1 \end{pmatrix}. \tag{8.77}$$

We must then employ a prescription analogous to the one presented below Definition 8.6.1 for electric-BSM contributions to avoid overcounting. This is again necessary since an electric-BSM wall and its metamorphic dual (8.77) have the same contribution to the index:

Proposition 8.6.2. *For a given set of charges (n, ℓ, m), the wall $\tilde{\gamma}$ has the same index contribution as γ to the polar coefficients $\widetilde{c}_m(n, \ell)$.*

Proof. This is proven completely analogously to the proof of Proposition 8.6.1. □

Furthermore we recall that, as explained below (8.32), the summand of the counting formula for negative discriminant states is invariant under an $\tilde{S}$-transformation. From (8.76), it is clear that the electric-BSM wall $\tilde{\gamma} \cdot \tilde{S}$ gives the same contribution to the polar coefficients $\widetilde{c}_m(n, \ell)$ as the magnetic-BSM wall γ. Thus, in addition to the above prescription that requires us to identify an electric-BSM wall with its metamorphic dual, we also need to identify the contribution of electric-BSM walls with the contribution of magnetic-BSM walls to avoid further overcounting. The BSM prescription in the case of magnetic or electric walls therefore identifies four contributions together for a given set of charges (n, ℓ, m).

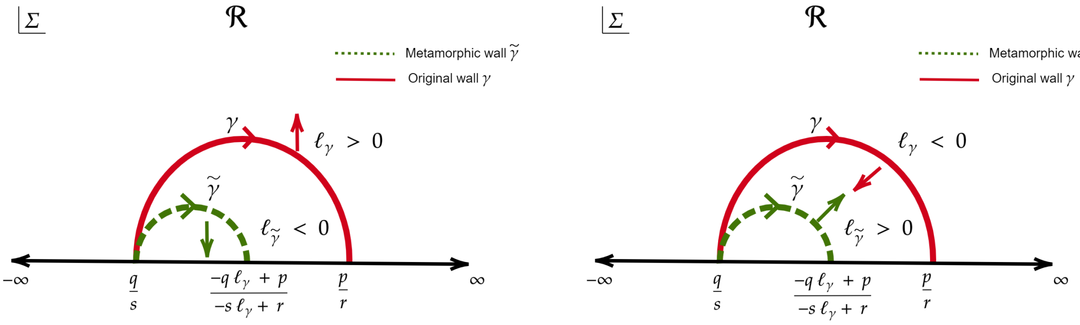
From Definition 8.6.2, it is easy to see that a wall γ and its metamorphic dual $\tilde{\gamma}$ have the same starting point q/s,

$$\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \iff \tilde{\gamma} = \begin{pmatrix} -q\ell_\gamma + p & q \\ -s\ell_\gamma + r & s \end{pmatrix}. \tag{8.78}$$

Given this and the fact that we look for walls in Γ_S^+ (with $p/r > q/s$), we have the following possible configurations for the electric-BSM wall γ and its dual:

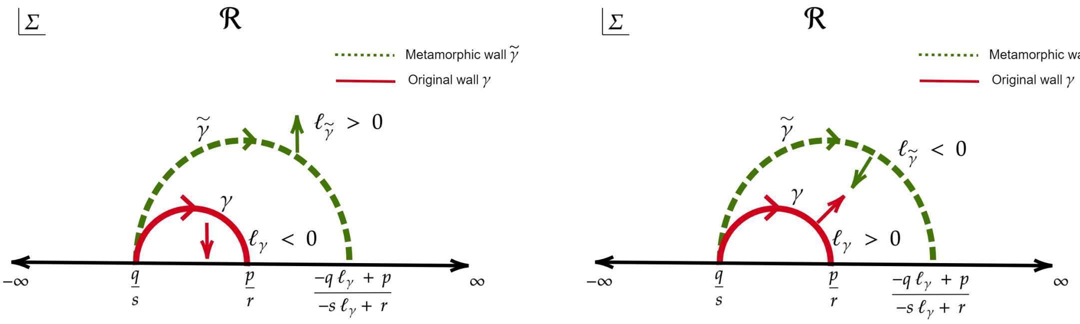
(1) **Case** $\dfrac{p}{r} > \dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r} > \dfrac{q}{s}$, as shown in Figure 15
   (a) For the situation $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$ as in Figure 15a, one can show that this configuration leads to a contradiction, analogous to the magnetic-BSM case of Figure 12a. Therefore this scenario does not occur.
   (b) The case $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$ as seen in Figure 15b does occur but does not contribute to the index in the region $\mathcal{R}$ owing to the BSM prescription.

(2) **Case** $\dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r} > \dfrac{p}{r} > \dfrac{q}{s}$, as shown in Figure 16



(A) Case $\dfrac{p}{r} > \dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r} > \dfrac{q}{s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$ (B) Case $\dfrac{p}{r} > \dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r} > \dfrac{q}{s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$

FIGURE 15. Metamorphosis for $\dfrac{p}{r} > \dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r} > \dfrac{q}{s}$



(A) Case $\dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r} > \dfrac{p}{r} > \dfrac{q}{s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$ (B) Case $\dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r} > \dfrac{p}{r} > \dfrac{q}{s}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$

FIGURE 16. Metamorphosis for $\dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r} > \dfrac{p}{r} > \dfrac{q}{s}$

(a) For the case $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$ as in 16a, there is again a contradiction which prevents this configuration from happening, as in the magnetic-BSM case of Figure 13a.

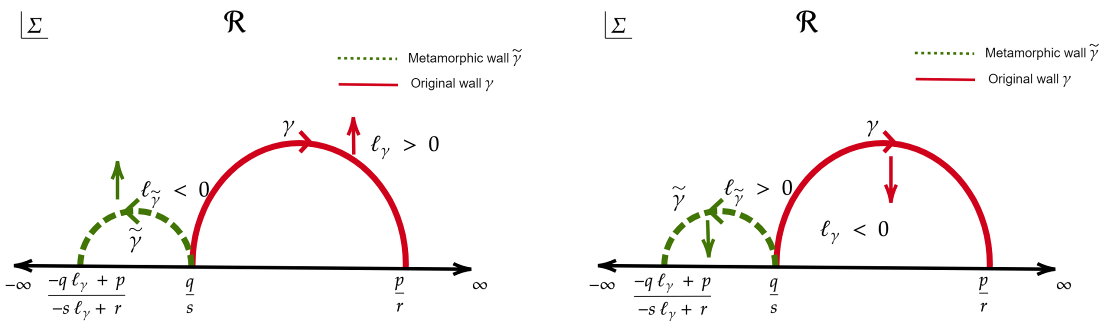
(b) The case $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$ as in Figure 16a does again occur but does not contribute to the index in the region $\mathcal{R}$ owing to the BSM prescription.

(3) **Case** $\dfrac{p}{r} > \dfrac{q}{s} > \dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r}$, as shown in Figure 17

(a) For the case as in Figure 17a, there will be a contribution to the index in the region $\mathcal{R}$ from γ and $\tilde{\gamma}$. Here, just as in the magnetic-BSM case, both these contributions are equal owing to Proposition 8.6.2 and must be identified according to the BSM prescription.

(b) The case as shown in Figure 17b does not contribute to the black hole degeneracy in the region $\mathcal{R}$ since neither γ nor $\tilde{\gamma}$ does.

Just as in the previous section, the above analysis shows that the only electric-BSM walls that give a non-trivial contribution to (8.36) must satisfy $m_\gamma \geq 0$, $n_\gamma =$



(A) Case $\dfrac{p}{r} > \dfrac{q}{s} > \dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} > 0$ (B) Case $\dfrac{p}{r} > \dfrac{q}{s} > \dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r}$, $\ell_\gamma = -\ell_{\tilde{\gamma}} < 0$

FIGURE 17. Metamorphosis for $\dfrac{p}{r} > \dfrac{q}{s} > \dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r}$

-1, $\ell_\gamma > 0$, as well as $\dfrac{q}{s} > \dfrac{-q\ell_\gamma + p}{-s\ell_\gamma + r}$. Once again, the last inequality leads to a stronger restriction on ℓ_γ, namely $\ell_\gamma > r/s$. We now explicitly give the form of the walls for which electric-BSM occurs, for all values of (n, ℓ, m) with the usual restrictions that $4mn - \ell^2 < 0$, $m > 0$ and $0 \le \ell \le m$. We make use of the observation at the beginning of this section regarding the action of $\tilde{S}$ on the metamorphic dual of a magnetic-BSM wall.

(1) **Case 1**: $m > 0$, $n = -1$
Acting on the metamorphic dual (8.59) of (8.69) with an $\tilde{S}$-transformation, we obtain the walls

$$\begin{pmatrix} 1 & 0 \\ s-\ell & 1 \end{pmatrix}, \quad \text{with} \quad \ell < s \le \frac{1}{2}\left(\ell + \sqrt{|\Delta|}\right) < m+1\,. \tag{8.79}$$

(2) **Case 2**: $m > 0$, $n = 0$ and $\ell > 0$
Acting on the metamorphic dual (8.59) of (8.71) with an $\tilde{S}$-transformation, we obtain the walls

$$\begin{pmatrix} \ell & 1 \\ m & \frac{m+1}{\ell} \end{pmatrix}, \quad \text{with} \quad \ell \,|\, (m+1)\,. \tag{8.80}$$

This set of matrices has entries that are bounded from above by $\max[m, (m+1)/\ell]$. Among all matrices that contribute to the index, we obtain here the maximal entry $m+1$ for $\ell = 1$.

(3) **Case 3**: $m > 0$, $n > 0$ and $\ell > 0$
Acting on the metamorphic dual (8.59) of the walls of Case 3 in Section 8.6.1 with an $\tilde{S}$-transformation, we obtain a finite set of electric-BSM walls. This can be shown analytically when acting on walls with $r = r_-$, and numerically when acting on walls with $r = r_+$. Moreover, all entries are strictly bounded from above by m. We refer the reader to Appendix C for some numerical checks.

Just as in the magnetic-BSM analysis of Section 8.6.1, in all the above cases we obtain a finite number of electric-BSM walls whose entries are bounded by $m+1$ (see

Equation (8.80)). In the next section, we turn to the final case that remains to be analyzed, which is when both transformed charges m_γ and n_γ are equal to -1.

8.6.3. Dyonic metamorphosis case: $m_\gamma = n_\gamma = -1$. The final case of metamorphosis occurs when both the electric and magnetic charges attain their lowest possible values. In the previous two cases of BSM, a magnetic or electric wall came with a single metamorphic dual, and as explained above the resulting four walls for a given charge vector (n, ℓ, m) have to be identified to obtain the correct contribution to the polar coefficients $\widetilde{c}_m(n, \ell)$. When both $m_\gamma = n_\gamma = -1$, there are two centers to be identified and we can identify the magnetic and electric centers alternatively. This generates an infinite sequence of dual walls [Sen11b]. The metamorphic duals can be generated in two ways depending on which center we start the identification with. Since they are equivalent, we choose to start the identification with the magnetic center.

Definition 8.6.3. *Let γ be a wall at which $m_\gamma = n_\gamma = -1$. The metamorphic duals are*

$$\tilde{\gamma}_i = \tilde{\gamma}_{i-1} \cdot M_{(i \bmod 2)} \quad \textit{for} \quad i > 0\,, \quad \textit{and} \quad \tilde{\gamma}_0 = \gamma\,, \tag{8.81}$$

where M_1, M_0 are defined as

$$M_1 \;\coloneqq\; \begin{pmatrix} 1 & -\ell_\gamma \\ 0 & 1 \end{pmatrix}, \qquad M_0 \;\coloneqq\; \begin{pmatrix} 1 & 0 \\ \ell_\gamma & 1 \end{pmatrix}. \tag{8.82}$$

For example, $\tilde{\gamma}_1 = \gamma \cdot M_1$, $\tilde{\gamma}_2 = \tilde{\gamma}_1 \cdot M_0$, $\tilde{\gamma}_3 = \tilde{\gamma}_2 \cdot M_1, \ldots$[9] Note that the identification of the electric center in M_0 does not have a '$-\ell_\gamma$' unlike in (8.77) and this dual wall will have the same sign of ℓ_γ as γ.

Proposition 8.6.3. *For a given set of charges (n, ℓ, m), the walls $\tilde{\gamma}_{i>0}$ all have the same index contribution as γ to the polar coefficients $\widetilde{c}_m(n, \ell)$.*

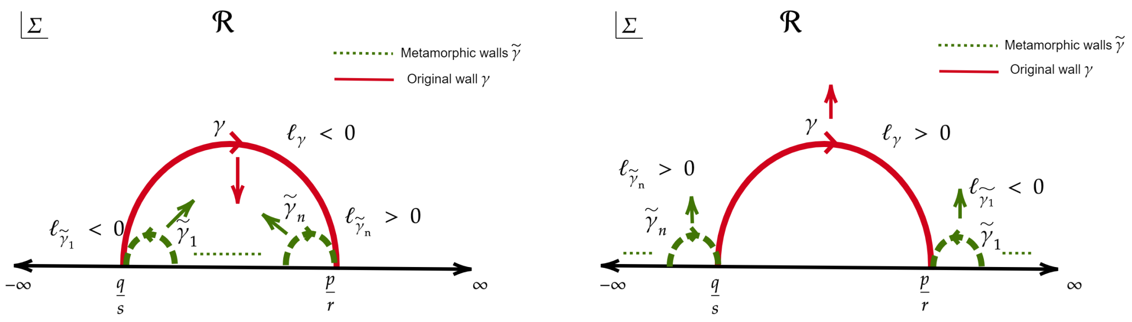
Proof. From the previous sections, we have already shown that the matrices that identify magnetic centers (8.59) and electric centers (8.77) leave the value of electric and magnetic charges invariant while only flipping the sign of ℓ_γ. Therefore, the infinite set of walls generated in (8.81) have the same contribution to the index. $\square$

We now characterize dyonic metamorphosis. The possible cases for dyonic metamorphosis are shown in Figure 18, where only one case as shown in Figure 18b can in principle contribute to the black hole degeneracy in the attractor region $\mathcal{R}$. The reason for this is the BSM prescription: all walls and their metamorphic must contribute in the same region to contribute to the polar coefficients $\widetilde{c}_m(n, \ell)$. To obtain the explicit form of the dyonic-BSM walls we must solve the following system,

$$\begin{aligned} n_\gamma &= s^2 n + q^2 m - sq\ell = -1\,, \\ m_\gamma &= r^2 n + p^2 m - rp\ell = -1\,, \\ \ell_\gamma &= -srn - pqm + \ell(ps + qr) = \sqrt{|\Delta| + 4}\,, \end{aligned} \tag{8.83}$$

with $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2, \mathbb{Z})$. It is important to recall that the discriminant Δ is a U-duality invariant. For this reason, the value of ℓ_γ is not independent and is fixed in

[9] Starting with the electric center, we would have $\tilde{\gamma}_1 = \gamma \cdot M_0$, $\tilde{\gamma}_2 = \tilde{\gamma}_1 \cdot M_1$, $\tilde{\gamma}_3 = \tilde{\gamma}_2 \cdot M_0, \ldots$



(A) Case of $m_\gamma = -1, n_\gamma = -1$ metamorphosis where all the metamorphic walls are inside the original wall.

(B) Case of $m_\gamma = -1, n_\gamma = -1$ metamorphosis where all the metamorphic walls are outside the original wall.

FIGURE 18. Possible cases of metamorphosis for $m_\gamma = -1, n_\gamma = -1$. There are an infinite series of walls to be identified but we have not depicted them here in order to avoid cluttering of the images.

terms of the charges (n, ℓ, m) as $\ell_\gamma^2 - 4 = \ell^2 - 4mn = |\Delta|$. We further restrict to the case where ℓ_γ is positive i.e., $\ell_\gamma = \sqrt{|\Delta| + 4}$ so that the wall contributes to the region $\mathcal{R}$. Given a charge vector (n, ℓ, m), there is an infinite sequence of walls, all with associated transformed charges $(n_\gamma,\ \ell_\gamma,\ m_\gamma) = (-1,\ \sqrt{|\Delta| + 4},\ -1)$, which get identified by the BSM prescription.

We now study the explicit form of the contributing walls. When $n = 0$, the discriminant is $|\Delta| = \ell^2$ (with $\ell > 0$). This reduces to (8.83) to

$$\begin{aligned} n_\gamma &= q^2 m - sq\ell \;=\; -1\,, \\ m_\gamma &= p^2 m - rp\ell \;=\; -1\,, \\ \ell_\gamma &= -pqm + \ell(ps + qr) \;=\; \sqrt{|\Delta| + 4}\,. \end{aligned} \tag{8.84}$$

Demanding that $\ell_\gamma \in \mathbb{Z}$ implies that $|\Delta| + 4$ is a perfect square i.e., $\ell^2 + 2^2 = \ell_\gamma^2$ for $\ell_\gamma \in \mathbb{Z}$. We know, however, that there is no Pythagorean triple with 2 as an element. (The difference of two squares form an increasing sequence $3, 5, 7, 8, \ldots$ and this does not include $2^2 = 4$.) Therefore, there is no dyonic metamorphosis for $n = 0$.

When $n \neq 0$, we can solve the system (8.83) after eliminating q using the $PSL(2, \mathbb{Z})$ relation $q = (ps - 1)/r$. From $m_\gamma = -1$ we obtain

$$r = r_\pm = \frac{1}{2n}\Big(\ell p \pm \sqrt{p^2|\Delta| - 4n}\Big)\,. \tag{8.85}$$

For each value of r, the condition $n_\gamma = -1$ is quadratic in s and yields two branches of solutions. We therefore arrive at the following dyonic-BSM walls[10]

(8.86)
$$\gamma_{+,\pm} = \begin{pmatrix} p & \frac{1}{2}\left(\pm p\sqrt{|\Delta|+4} + \sqrt{|\Delta|\, p^2 - 4n}\right) \\ \frac{1}{2n}\left(\ell p + \sqrt{p^2|\Delta| - 4n}\right) & \frac{1}{4n}\left(\ell p + \sqrt{|\Delta| p^2 - 4n}\right)\left(\ell \pm \sqrt{|\Delta|+4}\right) - mp \end{pmatrix},$$

and

(8.87)
$$\gamma_{-,\pm} = \begin{pmatrix} p & \frac{1}{2}\left(\pm p\sqrt{|\Delta|+4} - \sqrt{|\Delta|\, p^2 - 4n}\right) \\ \frac{1}{2n}\left(\ell p - \sqrt{p^2|\Delta| - 4n}\right) & \frac{1}{4n}\left(\ell p - \sqrt{|\Delta| p^2 - 4n}\right)\left(\ell \pm \sqrt{|\Delta|+4}\right) - mp \end{pmatrix}.$$

Since $-\gamma_{+,\pm}(-p) = \gamma_{-,\pm}(p)$ and we look for walls in $PSL(2,\mathbb{Z})$, we can focus on one type of walls, say $\gamma_{-,\pm}$. We therefore drop the first subscript and simply denote walls of the form (8.87) as $\gamma_\pm$. Examining the top-right entry of $\gamma_\pm$, a necessary condition for these walls to have integer entries is that $\sqrt{|\Delta|+4}, \sqrt{|\Delta| p^2 - 4n} \in \mathbb{Z}$. In the following we let $y = p$ and $D = |\Delta|$. The requirement that $D+4$ is a perfect square implies that D is not a square (as already observed above), and further that D is congruent to 0 or 1 modulo 4.

The requirement $\sqrt{Dy^2 - 4n} \in \mathbb{Z}$ can then be expressed as the requirement for y to be a solution of

$$\sqrt{D\, y^2 - 4n} \;=\; x \quad\Longrightarrow\quad x^2 - D\, y^2 \;=\; -4n\,, \tag{8.88}$$

with $x, y \in \mathbb{Z}$. We now split the discussion in two cases.

(1) **Case 1**: $n = -1$
In the case $n = -1$, the condition (8.88) takes the form

$$x^2 - D\, y^2 \;=\; 4\,. \tag{8.89}$$

This equation is the so-called Brahmagupta–Pell equation and has been well-studied over the years.[11] It is one of the classic Diophantine equations, and its solutions have been fully classified. In the language of modern algebraic number theory this problem is closely related to the problem of finding units in the ring of integers of the real quadratic field $\mathbb{Q}(\sqrt{D})$. We will present the solution below in elementary terms, and later make some comments on the more formal interpretation. We follow the treatment of [**Coh08, Con19b, Con19a**]. The equation (8.89) has an infinity of solutions given as follows. Let

$$u \;=\; u_0 + \sqrt{D}\, v_0\,, \tag{8.90}$$

be such that $u_0^2 - D\, v_0^2 = 4$ with the least strictly positive v_0. Then all solutions of (8.89) are given by [**Coh08**]

$$\frac{x + \sqrt{D}\, y}{2} \;=\; \left(\frac{u_0 + \sqrt{D}\, v_0}{2}\right)^k \quad \text{with} \quad k \in \mathbb{Z}\,. \tag{8.91}$$

[10] As mentioned above, a consequence of U-duality is that the equation $\ell_\gamma = \sqrt{|\Delta|+4}$ is not independent and does not yield additional constraints.

[11] In the literature, it is common to denote "the" Pell equation as the equation where the right-hand side is equal to one. However, the latter is a special case of (8.88) with $n = -1$, see e.g., [**Coh08, Con19b, Con19a**].

In general the difficulty is to find the fundamental solution u, as u_0 and v_0 need not be small even for small D.[12] In our case however, we can can use the physics of the problem which guarantees that $\ell_\gamma = \sqrt{D+4}$ is an integer. Therefore, the fundamental solution is simply

$$u = \sqrt{D+4} + \sqrt{D} \iff (u_0, v_0) = (\sqrt{D+4}, 1)\,. \tag{8.92}$$

From this solution we generate all other solutions from (8.91). Expanding that equation and matching the coefficients of unity and $\sqrt{D}$ leads to the recurrence

$$\begin{aligned} 2\,x_{k+1} &= \sqrt{D+4}\,x_k + D\,y_k\,, \\ 2\,y_{k+1} &= x_k + \sqrt{D+4}\,y_k\,, \end{aligned} \tag{8.93}$$

for $k \in \mathbb{Z}$. Given a solution $x_k + \sqrt{D}\,y_k$ to (8.89), the matrix (8.87) reads

$$\gamma_\pm(k) = \begin{pmatrix} y_k & \frac{1}{2}\Big(\pm y_k\sqrt{D+4} - x_k\Big) \\ \frac{1}{2}\big(x_k - \ell y_k\big) & \frac{1}{4}\Big(x_k\big(\ell \pm \sqrt{D+4}\big) - y_k\big(D \pm \ell\sqrt{D+4}\big)\Big) \end{pmatrix}. \tag{8.94}$$

Using the recursion relations (8.93), we can now show that acting on the right of $\gamma_+(k)$ with $M_1 \cdot \tilde{S}^{-1} = \begin{pmatrix} \sqrt{D+4} & 1 \\ -1 & 0 \end{pmatrix}$ yields

$$\gamma_+(k) \cdot M_1 \cdot \tilde{S}^{-1} = \gamma_+(k+1) \quad \forall\ k \in \mathbb{Z}\,, \tag{8.95}$$

while acting on the right of $\gamma_-(k)$ with $M_0 \cdot \tilde{S}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \sqrt{D+4} \end{pmatrix}$ yields

$$\gamma_-(k) \cdot M_0 \cdot \tilde{S}^{-1} = \gamma_-(k+1) \quad \forall\ k \in \mathbb{Z}\,. \tag{8.96}$$

Let us focus on $\gamma_+(k)$ [13]. In the language of Definition 8.6.3, the recurrence (8.95) can be written as

$$\tilde{\gamma}_k = \begin{cases} (-1)^{\frac{(|k|-1) \bmod 4}{2}} \gamma_+(k) \cdot P \cdot \tilde{S} & \text{for } |k| \geq 1 \text{ odd} \\ (-1)^{\frac{|k| \bmod 4}{2}} \gamma_+(k) \cdot P & \text{for } |k| \geq 2 \text{ even} \end{cases}, \quad \text{and} \quad \tilde{\gamma}_0 = \gamma_+(0) \cdot P\,, \tag{8.97}$$

where we used $M_0 = \tilde{S} \cdot M_1 \cdot \tilde{S}^{-1}$ and $P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Here, $k > 0$ corresponds to starting the identification with the magnetic center in Definition 8.6.3 and $k < 0$ corresponds to starting with the electric center. Equation (8.97) shows that all metamorphic duals of the dyonic-BSM walls $\gamma_+(0)$ are precisely all the solutions to the Brahmagupta–Pell equation (8.89).

The first representative of the orbit (the wall with $k = 0$) is given by

$$\gamma_+(0) = \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{2}\big(\ell + \sqrt{D+4}\big) \end{pmatrix}. \tag{8.98}$$

[12] A famous example (Fermat's challenge) is the equation $a^2 - D\,b^2 = 1$ with $D = 61$, where the fundamental solution is given by $a_0 = 1766319049$ and $b_0 = 226153980$. As we see below, this example does not appear in the physical system we study because $D + 4 = 65$ is not a perfect square.

[13] As $\gamma_-(0) = \gamma_+(1) \cdot \tilde{S}$, the metamorphic duals generated by $\gamma_-(k)$ are isomorphic to those generated by $\gamma_+(k)$.

Note that $\ell^2 = D-4m \equiv D+4 \,(\mathrm{mod}\, 2)$, which implies that $\ell \equiv \sqrt{D+4}\,(\mathrm{mod}\, 2)$, so that the bottom-right entry of (8.98) is always an integer. Therefore, provided that $\sqrt{D+4} \in \mathbb{Z}$, the wall $\gamma_+(0) \in \Gamma_S^+$ and all the metamorphic duals to be identified according to the BSM prescription are generated by the right action of $M_1 \cdot \tilde{S}^{-1}$. These dual walls are given by all the solutions to (8.89) as in (8.94). This completely characterizes dyonic-BSM in the case $n = -1$. Furthermore, since $0 \le \ell \le m$, the first representative of this orbit $\gamma_+(0)$ clearly has entries bounded by $0 < \frac{1}{2}\big(\ell + \sqrt{D+4}\big) \le m+1$.

(2) <u>**Case 2**</u>: $n \ge 1$
In this case we are interested in the solutions to the so-called generalized Brahmagupta–Pell equation (8.88)

$$x^2 - D\, y^2 \;=\; -4n\,. \tag{8.99}$$

As before, we have that $D > 0$ is not a square. Unlike in the $n = -1$ case, this equation does not necessarily have a solution for general D and n. However, when there is a solution (x_0, y_0) then there are infinitely many solutions which are all generated by multiplication with powers of the fundamental unit given in (8.92),

$$x + \sqrt{D}\, y \;=\; (x_0 + \sqrt{D}\, y_0)\left(\frac{\sqrt{D+4}+\sqrt{D}}{2}\right)^k \quad \text{for any } \; k \in \mathbb{Z}\,. \tag{8.100}$$

By repeating the same steps as in Case 1 above, one can again show that the orbit of metamorphic duals is precisely the solution set of the generalized Pell equation, and is generated by the matrices (8.95) and (8.96) acting on

$$\gamma_\pm(0) \;=\; \begin{pmatrix} y_0 & \frac{1}{2}\big(\pm y_0\sqrt{D+4} - x_0\big) \\ \frac{1}{2n}\big(\ell y_0 - x_0\big) & \frac{1}{4n}\big(\ell y_0 - x_0\big)\big(\ell \pm \sqrt{D+4}\big) - m y_0 \end{pmatrix}. \tag{8.101}$$

As in (8.98), we have $\sqrt{D+4} \in \mathbb{Z}$ and $\ell \equiv \sqrt{D+4}\ (\mathrm{mod}\, 2)$. In order for the matrix entries to be integer, a sufficient condition is $x_0 \equiv \ell y_0 \,(\mathrm{mod}\, 2n)$. By (8.99) we have $x_0^2 \equiv D\, y_0^2 \,(\mathrm{mod}\, 2n)$. Together with the fact that $D \equiv \ell^2 \,(\mathrm{mod}\, 2n)$, this implies that $x_0^2 \equiv \ell^2\, y_0^2 \,(\mathrm{mod}\, 2n)$, so that if n is square free, then we automatically have $x_0 \equiv \ell y_0 \,(\mathrm{mod}\, 2n)$. Once this condition is met, the full orbit of metamorphic duals is generated by $M_1 \cdot \tilde{S}^{-1}$ or $M_0 \cdot \tilde{S}^{-1}$ as before.

Since k runs over all integers in (8.100), it is clear that every Pell orbit—and therefore every dyonic BSM orbit—has an element with smallest $|y|$, which is called the fundamental solution. Although there is no existence theorem for solutions to the generalized Brahmagupta–Pell equation (8.99) when $n \ge 1$, there is a powerful theorem [**Con19b, Con19a**] which states that the fundamental solution is bounded according to

$$x^2 \;\le\; 2n\big(\sqrt{D+4}+\sqrt{D}\big)\,, \qquad y^2 \;\le\; 2n\left(\frac{\sqrt{D+4}+\sqrt{D}}{D}\right). \tag{8.102}$$

These bounds are very restrictive, and in particular, they imply that the set of dyonic-BSM orbits is finite, with a representative whose entries are strictly less than $m+1$.

We have thus fully characterized the dyonic-BSM walls and explained how the infinite orbit of metamorphic duals defined in Definition 8.6.3 is in one-to-one correspondence with the infinite orbit of solutions to the (generalized) Brahmagupta–Pell equation. Crucial to being able to solve the problem was the fact that U-duality fixes $\ell_\gamma = \sqrt{|\Delta| + 4}$ to be an integer.

It is instructive to restate the solutions of the Brahmagupta–Pell equation in the language of algebraic number theory [**Coh08, Con19b, Con19a**]. Consider the real quadratic field $K = \mathbb{Q}(\sqrt{D})$ where $D > 0$ is not a square. We will denote elements of this field either as (x, y) or as $x + \sqrt{D}\, y$ with $x, y \in \mathbb{Q}$. The norm of this element is $N(x, y) = x^2 - Dy^2$. By a change of variables ([**Coh08**], p. 355) one can bring the basic Brahmagupta–Pell equation to the form

$$x^2 - D\, y^2 \;=\; 1\,. \tag{8.103}$$

Thus we are looking for elements of norm 1. By multiplicativity of the norm, it is clear that if $u = x + \sqrt{D}\, y$ is a solution of (8.103), then so is u^k for $k \in \mathbb{Z}$. (It is easy to check, by rationalizing denominators and using (8.103), that negative powers are also good solutions.) The problem of finding all solutions to the basic Brahmagupta–Pell equation is then precisely the problem of finding all units in the order $\mathbb{Z}[\sqrt{D}]$. Denoting the discriminant of K as D_0, we have $D = D_0 f^2$. When $f = 1$ the solution to this problem is given by Dirichlet's unit theorem, that all solutions are generated as powers of the fundamental unit $a_0 + \sqrt{D}\, b_0$ which is the unit with least positive b_0. In fact this statement holds even when $f > 1$ (one can use a proof by induction on the number of prime powers of f). By changing variables back, we obtain the formulation (8.91).

For the general case we have, after the change of variables mentioned above,

$$x^2 - D\, y^2 \;=\; -n\,, \tag{8.104}$$

with $n \in \mathbb{Z}$ (We are however interested in $n \geq -1$ with $n \neq 0$). In this case, we are looking for elements in K with norm $-n$. Once again it is easy to see, by the multiplicativity of the norm, that given one such element (x_0, y_0) with $N(x_0, y_0) = -n$ we have an infinite number of elements with the same norm generated by multiplying $x_0 + \sqrt{D}\, y_0$ by arbitrary powers of a unit. The main theorem in this case says that there are a finite number of fundamental solutions (x_0, y_0) which lie in the range $|x_0| \leq \sqrt{|n|u}$, $|y_0| \leq \sqrt{|n|u/D}$, where u is any unit satisfying $u > 1$ and $N(u) = 1$. This last condition, translated back to the original variables is presented in (8.102).

Summary. For convenience, we now summarize the results of Sections 8.5 and 8.6 where we have characterized all the walls contributing to the negative discriminant degeneracies (8.36). There are two notable points.

Finiteness. Examining the various cases (with and without BSM), we see that the set of relevant walls is *finite*, and in fact small in the following sense: it consists of S–walls with entries bounded (in absolute value) from above by $m+1$, where the upper bound is optimal for certain values of the original charges (n, ℓ, m), as evidenced e.g., in (8.71). Moreover, all walls are such that $|p/r| \leq 1$ and $|q/s| \leq 1$ and so their endpoints lie in the strip $\Sigma_1 \in [-1, +1]$ in the Σ moduli space.

Structure. The structure of walls of electric and magnetic BSM form an orbit generated by the corresponding BSM transformation which acts as $\mathbb{Z}/2\mathbb{Z}$. The dyonic bound state metamorphosis has a very interesting characterization. We already knew that there is an infinite set of different-looking gravitational configurations, all with the same total dyonic charge invariants with negative discriminant, which are related by U-duality to each other. The phenomenon of BSM [**Sen11b, CLSS13**] says that these configurations actually should not be considered as distinct physical configurations; rather, they must be identified as different avatars of the same physical entity. The considerations in this section show that the following sets are in one-to-one correspondence:

1. The orbit of dyonic metamorphic duals with charges (n, ℓ, m) with $\ell^2 - 4mn = D > 0$, and

2a. Solutions to the generalized Brahmagupta–Pell equation $x^2 - D\,y^2 = -4n$ with fundamental solution (x_0, y_0), and the conditions $\sqrt{D+4} \in \mathbb{Z}$ and $x_0 \equiv \ell\, y_0 \pmod{2n}$, or, equivalently,

2b. The set of algebraic integers of norm $-n$ in the order $\mathbb{Z}[\sqrt{D}]$ of the real quadratic field $K = \mathbb{Q}(\sqrt{D})$ with $\frac{1}{2}(\sqrt{D+4} + \sqrt{D})$ as the fundamental unit, as well as the second congruence condition above.

Moreover, these sets are isomorphic to each other (and to $\mathbb{Z}$) as an additive group. The generators of the groups are given, respectively, by the generators in Definition 8.6.3 (modulo $\tilde{S}$), and by multiplication in K by the fundamental unit.

8.7. The exact black hole formula

In this section we assemble all the elements of the previous sections into one formula, and then we present checks of this formula. So far we have seen that the walls of marginal stability contributing to the polar coefficients according to Equation (8.36) are a subset of $PSL(2, \mathbb{Z})$. Bound state metamorphosis is an equivalence relation on the set $PSL(2, \mathbb{Z})$ and therefore divides it into orbits μ. We denote the set of orbits as (cf. Equation (8.34))

$$\Gamma_{\mathrm{BSM}}(n, \ell, m) \;=\; PSL(2, \mathbb{Z})/\mathrm{BSM}\,. \tag{8.105}$$

The orbits are of the following three types:

(1) Walls for which there are no metamorphosis. These walls have no duals and therefore lie in an orbit of length 1.
(2) Walls with either electric or magnetic metamorphosis, for which there is exactly one dual with the same contribution to the index. These walls lie in an orbit of length 2.
(3) Walls with dyonic metamorphosis for which there are an infinite number of dual walls. These walls lie in an orbit of infinite length with a group structure isomorphic to $\mathbb{Z}$.

We have seen that the contribution of an orbit to the index is one if all its elements contribute, and zero otherwise. This can be encoded in the following function defined on orbits (recalling Equations (8.30), (8.32) for the definition of the θ function),

$$\Theta(\mu) \;=\; \prod_{\gamma\in\mu} \theta(\gamma, \mathcal{R})\,, \qquad \mu \in \Gamma_{\mathrm{BSM}}(n, \ell, m)\,, \tag{8.106}$$

which can be lifted to a function on the space of walls as (using the same notation)

$$\Theta(\gamma) \;=\; \Theta(\mu)\,, \quad \gamma \in \mu\,. \tag{8.107}$$

We now have all the elements to present the full formula for the polar degeneracies (8.8). in the range $0 \le \ell \le m$ we have:[14]

$$\widetilde{c}_m(n,\ell) \;=\; \frac{1}{2} \sum_{\gamma \in \Gamma_{\mathrm{BSM}}(n,\ell,m)} (-1)^{\ell_\gamma+1}\,\Theta(\gamma)\,|\ell_\gamma|\,d(m_\gamma)\,d(n_\gamma)\,. \tag{8.108}$$

The sum in the above formula runs over $\Gamma_{\mathrm{BSM}}(n,\ell,m)$ which was defined as a coset of $PSL(2,\mathbb{Z})$ in (8.105). We can also write the formula so that the sum runs over a smaller set, by using the symmetry of the theory and making a choice in obtaining the coset representative. Such a choice makes the formula more explicit and is useful for computations. We had already illustrated the idea of two such formulas in the preliminary discussion in Section 8.4.2 where we didn't take BSM into account. In that case we had a sum over $PSL(2,\mathbb{Z})$ in (8.31) but by using the involution $\widetilde{S}$, we could equivalently write it as a sum over Γ_S^+ as in (8.33) with an additional factor of $\frac{1}{2}$. When BSM is present this discussion needs to be modified. When we have pure electric or pure magnetic BSM, the orbits of length 2 discussed in Case 2 above are actually part of a full symmetry orbit of length 4 via the following identifications:

$$(n_\gamma,\ell_\gamma,m_\gamma) \;=\; (N\neq -1, L>0,-1) \xmapsto{\widetilde{\gamma}_{\mathrm{m}}} (N,-L,-1) \xmapsto{\widetilde{S}} (-1,L,N) \xmapsto{\widetilde{\gamma}_{\mathrm{e}}} (-1,-L,N)\,. \tag{8.109}$$

In particular, the combined symmetry of BSM and $\widetilde{S}$ implies an identification of two walls in Γ_S^+, namely the first and the third of the above sequence.

By definition, a given wall belongs to one and only one symmetry orbit, and, as we have shown in the previous sections, when $0 \le \ell \le m$, every orbit has a non-empty intersection with the set

$$\left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \subset \Gamma_S^+ \;\middle|\; |p|\,,|q|\,,|r|\,,|s| \le m+1 \right\}. \tag{8.110}$$

The set $\mathrm{W}(n,\ell,m)$ is defined as the set of representative of orbits of BSM combined with $\widetilde{S}$ in this finite set having a non-zero value of Θ. With this definition we rewrite the degeneracies of negative discriminant states for $0 \le \ell \le m$ as

$$\widetilde{c}_m(n,\ell) \;=\; \sum_{\gamma \in \mathrm{W}(n,\ell,m)} (-1)^{\ell_\gamma+1}\,|\ell_\gamma|\,d(m_\gamma)\,d(n_\gamma)\,. \tag{8.111}$$

In Appendix C, we present a large amount of relevant data that supports the analysis and in particular shows that (8.108) corresponds to the degeneracies of negative discriminant states in $\psi_m^F(\tau,z)$.

8.8. Conclusions

In this chapter, we presented an exact formula for the count of negative discriminant states in (7.24) defined via (8.8). The exact formula is (8.108). This formula matches with the prediction of the Igusa cusp form and clears up an ambiguity with the localization answer as studied in [**MR16**]. The implications of this formula are as follows:

[14] Recall that the $\widetilde{c}_m(n,\ell)$ are coefficients of a (mock) Jacobi form of index m, and as such they are a function of $\Delta = 4mn-\ell^2$ and ℓ mod $(2m)$. Recall also that the modular properties imply that this can be further reduced to $0 \le \ell \le m$.

(a) The single centered black hole degeneracies are controlled by the degeneracies of the world sheet instantons by utilization of the Rademacher circle method for mock Jacobi forms (7.25) [**FR17**].
(b) Together with the (8.108) and the Rademacher technique, the extraction of the coefficients to reconstruct the Igusa cusp form can be made numerically faster.
(c) With regard to the world sheet instantons, the results that we exhibit here imply that the single center black hole degeneracy is controlled by the instanton partition function [**Nek03**] of only $m+1$ worldsheet instantons.
(d) The exact formula (8.108) can in principle be generalized for any chamber in the moduli space by careful redefinition of the theta function defined in (8.30).

A natural extension of this technique would have been to extend it to the cases of $\mathbb{Z}_N$ CHL orbifolds, for which the Rademacher technique has been studied from a black holes context in [**Nal19**]. This extension has been studied in [**CNR20**] for certain cases of N being a prime number.

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APPENDIX A

Superconformal characters

A.1. (Extended) $\mathcal{N}=2$ characters

For the (extended) $\mathcal{N}=2$ superconformal algebra with central charge $c=3d$, let $|\Omega\rangle$ denote a highest weight state with eigenvalues h,ℓ w.r.t. L_0 and J_0. Writing $\mathcal{H}_{h,\ell}$ for the representation belonging to $|\Omega\rangle$ we define the (graded) $\mathcal{N}=2$ characters in the Ramond sector through

$$\mathbf{ch}^{\mathcal{N}=2}_{d,h-\frac{c}{24},\ell}(\tau,z)=\mathrm{Tr}_{\mathcal{H}_{h,\ell}}((-1)^F q^{L_0-\frac{c}{24}}e^{2\pi i z J_0})\,, \tag{A.1}$$

where F is the fermion number and $q=e^{2\pi i\tau}$. Below we will also use $y=e^{2\pi i z}$. In the Ramond sector unitarity requires $h\geq\frac{c}{24}=\frac{d}{8}$.
The characters [**Oda89**, **Oda90b**, **Oda90a**] are given by (using the conventions of [**EH10a**][1]):

- Massles (BPS) representations exist for $h=\frac{d}{8}$; $\ell=\frac{d}{2},\frac{d}{2}-1,\frac{d}{2}-2,\ldots,-(\frac{d}{2}-1),-\frac{d}{2}$. For $\frac{d}{2}>\ell\geq 0$ they are given by

$$\mathbf{ch}^{\mathcal{N}=2}_{d,0,\ell\geq 0}(\tau,z)=(-1)^{\ell+\frac{d}{2}}\frac{(-i)\vartheta_1(\tau,z)}{\eta(\tau)^3}y^{\ell+\frac{1}{2}}\sum_{n\in\mathbb{Z}}q^{\frac{d-1}{2}n^2+(\ell+\frac{1}{2})n}\frac{(-y)^{(d-1)n}}{1-yq^n}, \tag{A.2}$$

  and for $\ell=\frac{d}{2}$ one has

$$\mathbf{ch}^{\mathcal{N}=2}_{d,0,\frac{d}{2}}(\tau,z)=(-1)^{d}\frac{(-i)\vartheta_1(\tau,z)}{\eta(\tau)^3}y^{\frac{d+1}{2}}\sum_{n\in\mathbb{Z}}q^{\frac{d-1}{2}n^2+\frac{d+1}{2}n}\frac{(1-q)\,(-y)^{(d-1)n}}{(1-yq^n)(1-yq^{n+1})}. \tag{A.3}$$

- Massive (non-BPS) representations exist for $h>\frac{d}{8}$; $\ell=\frac{d}{2},\frac{d}{2}-1,\ldots,-(\frac{d}{2}-1),-\frac{d}{2}$ and $\ell\neq 0$ for $d=$ even. For $\ell>0$ we have

$$\mathbf{ch}^{\mathcal{N}=2}_{d,h-\frac{c}{24},\ell>0}(\tau,z)=(-1)^{\ell+\frac{d}{2}}q^{h-\frac{d}{8}}\frac{i\vartheta_1(\tau,z)}{\eta(\tau)^3}y^{\ell-\frac{1}{2}}\sum_{n\in\mathbb{Z}}q^{\frac{d-1}{2}n^2+(\ell-\frac{1}{2})n}\,(-y)^{(d-1)n}\,. \tag{A.4}$$

In both cases the characters for $\ell<0$ are given by

$$\mathbf{ch}^{\mathcal{N}=2}_{d,h-\frac{c}{24},\ell<0}(\tau,z)=\mathbf{ch}^{\mathcal{N}=2}_{d,h-c/24,-\ell>0}(\tau,-z). \tag{A.5}$$

The Witten index of a massless representation is given by

$$\mathbf{ch}^{\mathcal{N}=2}_{d,0,\ell\geq 0}(\tau,z=0)=\begin{cases}(-1)^{\ell+\frac{d}{2}}, & \text{for } 0\leq\ell<\frac{\mathrm{d}}{2}\,,\\ 1+(-1)^d, & \text{for } \ell=\frac{\mathrm{d}}{2}\,.\end{cases} \tag{A.6}$$

[1] Note that our definition of $\vartheta_1(\tau,z)$ differs by a minus sign from the definition used there.

A.2. $\mathcal{N}=4$ characters

Analogously to the $\mathcal{N}=2$ case the (graded) characters of the $\mathcal{N}=4$ superconformal algebra with central charge $c=3d$, and d even, in the Ramond sector are defined as

$$\mathbf{ch}^{\mathcal{N}=4}_{d,h-\frac{c}{24},\ell}(\tau,z)=\mathrm{Tr}_{\mathcal{H}_{h,\ell}}((-1)^F q^{L_0-\frac{c}{24}}e^{4\pi i z T_0^3}),\tag{A.7}$$

where h and ℓ are the eigenvalues of L_0 and T_0^3 of the highest weight state belonging to the representation $\mathcal{H}_{h,\ell}$. As in the $\mathcal{N}=2$ case unitarity requires $h\geq\frac{d}{8}$.
The characters [**ET88a**] are given by (using conventions from [**EH10b**])

- Massless representation exist for $h=\frac{d}{8}$, $\ell=0,\frac{1}{2}\dots,\frac{d}{4}$ and are given by

$$\mathbf{ch}^{\mathcal{N}=4}_{d,0,\ell}(\tau,z)=\frac{i}{\vartheta_1(\tau,2z)}\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^3}\sum_{\varepsilon=\pm1}\sum_{m\in\mathbb{Z}}\varepsilon\frac{e^{4\pi i\varepsilon((\frac{d}{2}+1)m+\ell)(z+\frac{1}{2})}}{(1-y^{-\varepsilon}q^{-m})^2}q^{(\frac{d}{2}+1)m^2+2\ell m}.\tag{A.8}$$

In particular for $\ell=0$ this may be written as

$$\mathbf{ch}^{\mathcal{N}=4}_{d,0,0}(\tau,z)=\frac{-i}{\vartheta_1(\tau,2z)}\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^3}\sum_{m\in\mathbb{Z}}q^{(\frac{d}{2}+1)m^2}y^{(d+2)m}\frac{1+yq^m}{1-yq^m}.\tag{A.9}$$

- Massive representation exist for $h>\frac{d}{8}$, $\ell=\frac{1}{2},1,\dots,\frac{d}{4}$ and are given by

$$\mathbf{ch}^{\mathcal{N}=4}_{d,h-\frac{c}{24},\ell}(\tau,z)=iq^{h-\frac{2\ell^2}{d+2}-\frac{d}{8}}\frac{\vartheta_1(\tau,z)^2}{\vartheta_1(\tau,2z)\eta(\tau)^3}\left(\vartheta_{\frac{d}{2}+1,2\ell}(\tau,z+\frac{1}{2})-\vartheta_{\frac{d}{2}+1,-2\ell}(\tau,z+\frac{1}{2})\right),\tag{A.10}$$

where

$$\vartheta_{P,a}(\tau,z)=\sum_{n\in\mathbb{Z}}q^{\frac{(2Pn+a)^2}{4P}}y^{2Pn+a}\,.\tag{A.11}$$

With the help of the $\mathcal{N}=4$ characters combinations of massless $\mathcal{N}=2$ characters which are even in z can be expressed in the following way

$$\mathbf{ch}^{\mathcal{N}=2}_{d,0,\frac{1}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{d,0,-\frac{1}{2}}(\tau,z)=(-1)^{\frac{d+1}{2}}\phi_{0,\frac{3}{2}}(\tau,z)\mathbf{ch}^{\mathcal{N}=4}_{d-3,0,0}(\tau,z)\,,\tag{A.12}$$

$$\mathbf{ch}^{\mathcal{N}=2}_{d,0,\frac{3}{2}}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{d,0,-\frac{3}{2}}(\tau,z)=(-1)^{\frac{d+1}{2}}\phi_{0,\frac{3}{2}}(\tau,z)\left(\mathbf{ch}^{\mathcal{N}=4}_{d-3,0,0}(\tau,z)+\mathbf{ch}^{\mathcal{N}=4}_{d-3,0,\frac{1}{2}}(\tau,z)\right).\tag{A.13}$$

Likewise the even-z combination of the massive $\mathcal{N}=2$ characters can be written as

$$\mathbf{ch}^{\mathcal{N}=2}_{d,n,l}(\tau,z)+\mathbf{ch}^{\mathcal{N}=2}_{d,n,-l}(\tau,z)=(-1)^{2l+\frac{d-1}{2}}\phi_{0,\frac{3}{2}}(\tau,z)\mathbf{ch}^{\mathcal{N}=4}_{d-3,n,\frac{1}{2}(l-\frac{1}{2})}(\tau,z)\,.\tag{A.14}$$

APPENDIX B

Character table of $\mathbf{M}_{12}$ and $\mathbf{M}_{24}$

TABLE 3. The character table of M_{12} where we use the notation $e_{11} = \frac{1}{2}(-1 + i\sqrt{11})$.

$[g]$	1a	2a	2b	3a	3b	4a	4b	5a	6a	6b	8a	8b	10a	11a	11b
$[g^2]$	1a	1a	1a	3a	3b	2b	2b	5a	3b	3a	4a	4b	5a	11b	11a
$[g^3]$	1a	2a	2b	1a	1a	4a	4b	5a	2a	2b	8a	8b	10a	11a	11b
$[g^5]$	1a	2a	2b	3a	3b	4a	4b	1a	6a	6b	8a	8b	2a	11a	11b
$[g^{11}]$	1a	2a	2b	3a	3b	4a	4b	5a	6a	6b	8a	8b	10a	1a	1a
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	11	-1	3	2	-1	-1	3	1	-1	0	-1	1	-1	0	0
χ_3	11	-1	3	2	-1	3	-1	1	-1	0	1	-1	-1	0	0
χ_4	16	4	0	-2	1	0	0	1	1	0	0	0	-1	e_{11}	$\bar{e}_{11}$
χ_5	16	4	0	-2	1	0	0	1	1	0	0	0	-1	$\bar{e}_{11}$	e_{11}
χ_6	45	5	-3	0	3	1	1	0	-1	0	-1	-1	0	1	1
χ_7	54	6	6	0	0	2	2	-1	0	0	0	0	1	-1	-1
χ_8	55	-5	7	1	1	-1	-1	0	1	1	-1	-1	0	0	0
χ_9	55	-5	-1	1	1	3	-1	0	1	-1	-1	1	0	0	0
χ_{10}	55	-5	-1	1	1	-1	3	0	1	-1	1	-1	0	0	0
χ_{11}	66	6	2	3	0	-2	-2	1	0	-1	0	0	1	0	0
χ_{12}	99	-1	3	0	3	-1	-1	-1	-1	0	1	1	-1	0	0
χ_{13}	120	0	-8	3	0	0	0	0	0	1	0	0	0	-1	-1
χ_{14}	144	4	0	0	-3	0	0	-1	1	0	0	0	-1	1	1
χ_{15}	176	-4	0	-4	-1	0	0	1	-1	0	0	0	1	0	0

TABLE 4. The character table of M_{24} with shorthand notation $e_n = \frac{1}{2}(-1 + \mathrm{i}\sqrt{n})$.

[g]	1a	2a	2b	3a	3b	4a	4b	4c	5a	6a	6b	7a	7b	8a	10a	11a	12a	12b	14a	14b	15a	15b	21a	21b	23a	23b
$[g^2]$	1a	1a	1a	3a	3b	2a	2a	2b	5a	3a	3b	7a	7b	4b	5a	11a	6a	6b	7a	7b	15a	15b	21a	21b	23a	23b
$[g^3]$	1a	2a	2b	1a	1a	4a	4b	4c	5a	2a	2b	7b	7a	8a	10a	11a	4a	4c	14b	14a	5a	5a	7b	7a	23a	23b
$[g^5]$	1a	2a	2b	3a	3b	4a	4b	4c	1a	6a	6b	7b	7a	8a	2b	11a	12a	12b	14b	14a	3a	3a	21b	21a	23b	23a
$[g^7]$	1a	2a	2b	3a	3b	4a	4b	4c	5a	6a	6b	1a	1a	8a	10a	11a	12a	12b	2a	2a	15b	15a	3b	3b	23b	23a
$[g^{11}]$	1a	2a	2b	3a	3b	4a	4b	4c	5a	6a	6b	7a	7b	8a	10a	1a	12a	12b	14a	14b	15b	15a	21a	21b	23b	23a
$[g^{23}]$	1a	2a	2b	3a	3b	4a	4b	4c	5a	6a	6b	7a	7b	8a	10a	11a	12a	12b	14a	14b	15a	15b	21a	21b	1a	1a
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2	1	-1	1	-1	-1	0	0	0	0	-1	-1	0	0
χ_3	45	-3	5	0	3	-3	1	1	0	0	-1	e_7	$\bar{e}_7$	-1	0	1	0	1	$-e_7$	$-\bar{e}_7$	0	0	e_7	$\bar{e}_7$	-1	-1
χ_4	45	-3	5	0	3	-3	1	1	0	0	-1	$\bar{e}_7$	e_7	-1	0	1	0	1	$-\bar{e}_7$	$-e_7$	0	0	$\bar{e}_7$	e_7	-1	-1
χ_5	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	e_{15}	$\bar{e}_{15}$	0	0	1	1
χ_6	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	$\bar{e}_{15}$	e_{15}	0	0	1	1
χ_7	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1
χ_8	253	13	-11	10	1	-3	1	1	3	-2	1	1	1	-1	-1	0	0	1	-1	-1	0	0	1	1	0	0
χ_9	483	35	3	6	0	3	3	3	-2	2	0	0	0	-1	-2	-1	0	0	0	0	1	1	0	0	0	0
χ_{10}	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	e_{23}	$\bar{e}_{23}$
χ_{11}	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	$\bar{e}_{23}$	e_{23}
χ_{12}	990	-18	-10	0	3	6	2	-2	0	0	-1	e_7	$\bar{e}_7$	0	0	0	0	1	e_7	$\bar{e}_7$	0	0	e_7	$\bar{e}_7$	1	1
χ_{13}	990	-18	-10	0	3	6	2	-2	0	0	-1	$\bar{e}_7$	e_7	0	0	0	0	1	$\bar{e}_7$	e_7	0	0	$\bar{e}_7$	e_7	1	1
χ_{14}	1035	27	35	0	6	3	-1	3	0	0	2	-1	-1	1	0	1	0	0	-1	-1	0	0	-1	-1	0	0
χ_{15}	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2e_7$	$2\bar{e}_7$	-1	0	1	0	-1	0	0	0	0	$-e_7$	$-\bar{e}_7$	0	0
χ_{16}	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2\bar{e}_7$	$2e_7$	-1	0	1	0	-1	0	0	0	0	$-\bar{e}_7$	$-e_7$	0	0
χ_{17}	1265	49	-15	5	8	-7	1	-3	0	1	0	-2	-2	1	0	0	-1	0	0	0	0	0	1	1	0	0
χ_{18}	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0	-1	1	0	0	-1	0	0	1	1	0	0	0	0
χ_{19}	2024	8	24	-1	8	8	0	0	-1	-1	0	1	1	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0
χ_{20}	2277	21	-19	0	6	-3	1	-3	-3	0	2	2	2	-1	1	0	0	0	0	0	0	0	-1	-1	0	0
χ_{21}	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1	0	1	1	0	0	-1	-1	0	0	1	1	0	0
χ_{22}	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1	0	0	0	0	0	1	1	0	0	-1	-1	1	1
χ_{23}	5313	49	9	-15	0	1	-3	-3	3	1	0	0	0	-1	-1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	5544	-56	24	9	0	-8	0	0	-1	1	0	0	0	0	-1	0	1	0	0	0	-1	-1	0	0	1	1
χ_{25}	5796	-28	36	-9	0	-4	4	0	1	-1	0	0	0	0	1	-1	-1	0	0	0	1	1	0	0	0	0
χ_{26}	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	-1

APPENDIX C

Check of degeneracies

We now present checks of (8.108). Table 5 lists negative discriminant states for $m = 1, \ldots, 5$. Column I lists the charge invariants (m, n, ℓ) with discriminant $\Delta = 4mn - \ell^2 < 0$. Note that we have changed the order of the charge invariants here with respect to the rest of the chapter. The organization is as follows: we first list m which is the index of the mock Jacobi form, followed by n and ℓ. The range of ℓ is $m, \ldots, 0$ which covers all the cases and n runs over all values that produce a negative discriminant with non-zero coefficient for ψ_m^{F}. Column II lists the walls $\gamma \in \mathrm{W}(n, \ell, m)$ which contribute to the degeneracy of states with these charge invariants. These walls have been discussed in Sections 8.5 and 8.6. The walls in Column II, as stated in (8.27), are semicircles from $q/s \to p/r$, where $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is a $PSL(2,\mathbb{Z})$ matrix. Column III shows the transformed charges at the wall γ. In the γ-transformed S-duality frame the decay products are $(Q_\gamma, 0)$ and $(0, P_\gamma)$ with invariants $(m_\gamma, n_\gamma, \ell_\gamma)$ (cf. (8.24) and (8.26)). Cases with either $m_\gamma = -1$ or $n_\gamma = -1$ correspond, respectively, to magnetic and electric metamorphosis. An example is $(m, n, \ell) = (1, -1, 0)$ where we have the walls $(m_\gamma, n_\gamma, \ell_\gamma) = (-1, 0, 2)$ and $(-1, 0, -2)$ (with contribution 48) are identified due to magnetic BSM as shown in the table. According to the discussion around (8.109), we also need to identify these walls with $(0, -1, 2)$ and $(0, -1, -2)$ (which we have not explicitly displayed in the table). In a similar manner, we have only displayed pure magnetic, but not pure electric BSM phenomena in the table. The cases with $m_\gamma = n_\gamma = -1$ correspond to dyonic metamorphosis, in which case an infinite number of walls must be identified (see Section 8.6.3). An example is $(m, n, \ell) = (1, -1, 1)$. Here we have exhibited four walls corresponding to the first two solutions to the Brahmagupta–Pell equation (8.89) (the trivial solution with $p = 0$ and the first non-trivial one with $p = 1$) and their respective first metamorphic duals ($\tilde{\gamma}$ built with M_1 in Definition 8.6.3). Column IV is the index contribution of each wall and Column V is the total index $\widetilde{c}_m(n, \ell)$ according to (8.111). This agrees with with a direct calculation of the polar degeneracies of $\psi_m^{\mathrm{F}}(\tau, z)$.

I. **Charges** $(m, n, \ell\,; \Delta)$	**II.** **Walls** $\gamma = q/s \to p/r$	**III.** **Transf. charges** $(m_\gamma,\ n_\gamma,\ \ell_\gamma)$	**IV.** **Contribution from wall**	**V.** **Net Index** $\widetilde{c}_m(n, \ell)$
$(1, -1, 1\,; -5)$	$-1/2 \to 0/1$ $-1/-1 \to 0/1$ $0/1 \to 1/1$ $-3/-2 \to 1/1$ $\vdots$	$(-1,\ -1,\ 3)$ $(-1,\ -1,\ -3)$ $(-1,\ -1,\ 3)$ $(-1,\ -1,\ -3)$ $\vdots$	3	3
$(1, -1, 0\,; -4)$	$-1/1 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 0,\ 2)$ $(-1,\ 0,\ -2)$	48	48
$(1, 0, 1\,; -1)$	$0/1 \to 1/1$	$(0,\ 0,\ 1)$	576	600
	$0/1 \to 1/2$ $-1/-1 \to 1/2$	$(-1, 0,\ 1)$ $(-1, 0,\ -1)$	24	
$(2, -1, 2\,; -12)$	$-1/3 \to 0/1$ $-1/-1 \to 0/1$ $0/1 \to 1/1$ $-4/-3 \to 1/1$ $\vdots$	$(-1,\ -1,\ 4)$ $(-1,\ -1,\ -4)$ $(-1,\ -1,\ 4)$ $(-1,\ -1,\ -4)$ $\vdots$	4	4
$(2, -1, 1\,; -9)$	$-1/2 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 0,\ 3)$ $(-1,\ 0,\ -3)$	72	72
$(2, -1, 0\,; -8)$	$-1/1 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 1,\ 2)$ $(-1,\ 1,\ -2)$	648	648
$(2, 0, 2\,; -4)$	$0/1 \to 1/1$	$(0,\ 0,\ 2)$	1152	1152
$(2, 0, 1\,; -1)$	$0/1 \to 1/1$	$(1,\ 0,\ 1)$	7776	8376
	$0/1 \to 1/2$	$(0,\ 0,\ 1)$	576	
	$0/1 \to 1/3$ $-1/-2 \to 1/3$	$(-1, 0,\ 1)$ $(-1, 0,\ -1)$	24	
$(3, -1, 3\,; -21)$	$-1/4 \to 0/1$ $-1/-1 \to 0/1$ $0/1 \to 1/1$ $-5/-4 \to 1/1$ $\vdots$	$(-1,\ -1,\ 5)$ $(-1,\ -1,\ -5)$ $(-1,\ -1,\ 5)$ $(-1,\ -1,\ -5)$ $\vdots$	5	5
$(3, -1, 2\,; -16)$	$-1/3 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 0,\ 4)$ $(-1,\ 0,\ -4)$	96	96
$(3, -1, 1\,; -13)$	$-1/2 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 1,\ 3)$ $(-1,\ 1,\ -3)$	972	972
$(3, -1, 0\,; -12)$	$-1/1 \to 0/1$ $-1/-1 \to 0/1$	$(-1, 2,\ 2)$ $(-1, 2,\ -2)$	6400	6404
	$-1/2 \to 0/1$ $-1/-2 \to 0/1$ $0/1 \to 1/2$ $-4/-7 \to 1/2$ $\vdots$	$(-1, -1,\ 4)$ $(-1, -1,\ -4)$ $(-1, -1,\ 4)$ $(-1, -1,\ -4)$ $\vdots$	4	
$(3, 0, 3\,; -9)$	$0/1 \to 1/1$	$(0,\ 0,\ 3)$	1728	1728
$(3, 0, 2\,; -4)$	$0/1 \to 1/1$	$(1,\ 0,\ 2)$	15552	15600
	$0/1 \to 1/2$	$(-1,\ 0,\ 2)$	48	

	$-2/-3 \to 1/2$	$(-1,\ 0,\ -2)$		
$(3, 0, 1\,; -1)$	$0/1 \to 1/1$	$(2,\ 0,\ 1)$	76800	85176
	$0/1 \to 1/2$	$(1,\ 0,\ 1)$	7776	
	$0/1 \to 1/3$	$(0,\ 0,\ 1)$	576	
	$0/1 \to 1/4$ $-1/-3 \to 1/4$	$(-1, 0,\ 1)$ $(-1, 0,\ -1)$	24	
$(4, -1, 4\,; -32)$	$-1/5 \to 0/1$ $-1/-1 \to 0/1$ $0/11 \to 1/1$ $-6/-5 \to 1/1$ $\vdots$	$(-1,\ -1,\ 6)$ $(-1,\ -1,\ -6)$ $(-1,\ -1,\ 6)$ $(-1,\ -1,\ -6)$ $\vdots$	6	6
$(4, -1, 3\,; -25)$	$-1/4 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 0,\ 5)$ $(-1,\ 0,\ -5)$	120	120
$(4, -1, 2\,; -20)$	$-1/3 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 1,\ 4)$ $(-1,\ 1,\ -4)$	1296	1296
$(4, -1, 1\,; -25)$	$-1/2 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 2,\ 3)$ $(-1,\ 2,\ -3)$	9600	9600
$(4, -1, 0\,; -16)$	$-1/1 \to 0/1$ $-1/-1 \to 0/1$	$(-1, 3,\ 2)$ $(-1, 3,\ -2)$	51300	51396
	$-1/2 \to 0/1$ $-1/-2 \to 0/1$	$(-1,\ 0,\ 4)$ $(-1,\ 0,\ -4)$	96	
$(4, 0, 4\,; -16)$	$0/1 \to 1/1$	$(0,\ 0,\ 4)$	2304	2304
$(4, 0, 3\,; -9)$	$0/1 \to 1/1$	$(1,\ 0,\ 3)$	23328	23328
$(4, 0, 2\,; -4)$	$0/1 \to 1/1$	$(2,\ 0,\ 2)$	153600	154752
	$0/1 \to 1/2$	$(0,\ 0,\ 2)$	1152	
$(4, 0, 1\,; -1)$	$0/1 \to 1/1$	$(3,\ 0,\ 1)$	615600	700776
	$0/1 \to 1/2$	$(2,\ 0,\ 1)$	76800	
	$0/1 \to 1/3$	$(1,\ 0,\ 1)$	7776	
	$0/1 \to 1/4$	$(0,\ 0,\ 1)$	576	
	$0/1 \to 1/5$ $-1/-4 \to 1/5$	$(-1, 0,\ 1)$ $(-1, 0,\ -1)$	24	
$(5, -1, 5\,; -45)$	$-1/6 \to 0/1$ $-1/-1 \to 0/1$ $0/1 \to 1/1$ $-7/-6 \to 1/1$ $\vdots$	$(-1,\ -1,\ 7)$ $(-1,\ -1,\ -7)$ $(-1,\ -1,\ 7)$ $(-1,\ -1,\ -7)$ $\vdots$	7	7
$(5, -1, 4\,; -36)$	$-1/5 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 0,\ 6)$ $(-1,\ 0,\ -6)$	144	144
$(5, -1, 3\,; -29)$	$-1/4 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 1,\ 5)$ $(-1,\ 1,\ -5)$	1620	1620
$(5, -1, 2\,; -24)$	$-1/3 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 2,\ 4)$ $(-1,\ 2,\ -4)$	12800	12800
$(5, -1, 1\,; -21)$	$-1/2 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 3,\ 3)$ $(-1,\ 3,\ -3)$	76950	76955
	$-1/3 \to 0/1$ $-1/-2 \to 0/1$ $0/1 \to 1/2$	$(-1,\ -1,\ 5)$ $(-1,\ -1,\ -5)$ $(-1,\ -1,\ 5)$	5	

	$-5/-9 \to 1/2$	$(-1,\ -1,\ -5)$		
	$\vdots$	$\vdots$		
$(5,-1,0\,;-20)$	$-1/1 \to 0/1$	$(-1,\ 4,\ 2)$	352512	353808
	$-1/-1 \to 0/1$	$(-1,\ 4,\ -2)$		
	$-1/2 \to 0/1$	$(-1,\ 1,\ 4)$	1296	
	$-1/-2 \to 0/1$	$(-1,\ 1,\ -4)$		
$(5,0,5\,;-25)$	$0/1 \to 1/1$	$(0,\ 0,\ 5)$	2880	2880
$(5,0,4\,;-16)$	$0/1 \to 1/1$	$(1,\ 0,\ 4)$	31104	31104
$(5,0,3\,;-9)$	$0/1 \to 1/1$	$(2,\ 0,\ 3)$	230400	230472
	$0/1 \to 1/2$	$(-1,\ 0,\ 3)$	72	
	$-3/-5 \to 1/2$	$(-1,\ 0,\ -3)$		
$(5,0,2\,;-4)$	$0/1 \to 1/1$	$(3,\ 0,\ 2)$	1231200	1246800
	$0/1 \to 1/2$	$(1,\ 0,\ 2)$	15552	
	$0/1 \to 1/3$	$(-1,\ 0,\ 2)$	48	
	$-2/-5 \to 1/3$	$(-1,\ 0,\ -2)$		
$(5,0,1\,;-1)$	$0/1 \to 1/1$	$(4,\ 0,\ 1)$	4230144	4930920
	$0/1 \to 1/2$	$(3,\ 0,\ 1)$	615600	
	$0/1 \to 1/3$	$(2,\ 0,\ 1)$	76800	
	$0/1 \to 1/4$	$(1,\ 0,\ 1)$	7776	
	$0/1 \to 1/5$	$(0,\ 0,\ 1)$	576	
	$0/1 \to 1/6$	$(-1,\ 0,\ 1)$	24	
	$-1/-5 \to 1/6$	$(-1,\ 0,\ -1)$		
$(5,1,5\,;-5)$	$0/1 \to 1/1$	$(1,\ 1,\ 3)$	314928	315255
	$0/1 \to 1/2$	$(-1,\ 1,\ 1)$	324	
	$-1/-1 \to 1/2$	$(-1,\ 1,\ -1)$		
	$1/3 \to 1/2$	$(-1,\ -1,\ 3)$	3	
	$-2/-3 \to 1/2$	$(-1\ -1,\ -3)$		
	$1/2 \to 2/3$	$(-1,\ -1,\ 3)$		
	$-5/-7 \to 2/3$	$(-1,\ -1,\ -3)$		
	$\vdots$	$\vdots$		

TABLE 5. Table of examples detailing original charge vector, contributing walls, associated charge breakdowns at walls and index contributions.

We now present more data to establish agreement between the formula (8.111) and the polar coefficients of ψ^{F}_m. The first column in the table below gives the value of (m,n,ℓ) charges while the second column shows the degeneracy (corresponding to Columns I and V of Table 5).

(m,n,ℓ)	Degeneracy
(8,-1,-8)	10
(8,-1,-7)	216
(8,-1,-6)	2592
(8,-1,-5)	22400
(8,-1,-4)	153900
(8,-1,-3)	881280
(8,-1,-2)	4295024
(8,-1,-1)	17807488
(8,-1,0)	61062180
(8,0,-8)	4608
(8,0,-7)	54432
(8,0,-6)	460800
(8,0,-5)	3078000
(8,0,-4)	16922880
(8,0,-3)	77538312
(8,0,-2)	293278848
(8,0,-1)	897317904
(9,-1,-9)	11
(9,-1,-8)	240
(9,-1,-7)	2916
(9,-1,-6)	25600
(9,-1,-5)	179550
(9,-1,-4)	1057536
(9,-1,-3)	5368607
(9,-1,-2)	23723928
(9,-1,-1)	90663975
(9,-1,0)	290663024
(9,0,-9)	5184
(9,0,-8)	62208
(9,0,-7)	537600
(9,0,-6)	3693600
(9,0,-5)	21150840
(9,0,-4)	103108224
(9,0,-3)	428844240
(9,0,-2)	1501356960
(9,0,-1)	4333733904
(10,-1,-10)	12
(10,-1,-9)	264
(10,-1,-8)	3240
(10,-1,-7)	28800
(10,-1,-6)	205200
(10,-1,-5)	1233792
(10,-1,-4)	6442320
(10,-1,-3)	29652648
(10,-1,-2)	120733500
(10,-1,-1)	430433280
(10,-1,0)	1302222528
(10,0,-10)	5760
(10,0,-9)	69984
(10,0,-8)	614400
(10,0,-7)	4309200
(10,0,-6)	25380864
(10,0,-5)	128849280
(10,0,-4)	569634816
(10,0,-3)	2185571160
(10,0,-2)	7166110848
(10,0,-1)	19675717080
(11,-1,-11)	13
(11,-1,-10)	288
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Curriculum Vita

Abhiram Mamandur Kidambi

Contact

Address: Institute for Theoretical Physics, Technical University of Vienna, Wiedner Hauptstraße 8-10, 1040 Vienna Austria &
Stanford Institue for theoretical Physics, Stanford Univeristy, 382 Via Pueblo Mall, Stanford, CA, USA
Email: mkabhiram@gmail.com
url: abhirammk.github.io

Personal

Date of Birth: Sept. 4 1991
Place of Birth: Bangalore, India
Nationality: Indian
Languages: English, Tamil, Kannada, Hindi/Urdu (Native), Sanskrit, German (Full Professional), Spanish, Telugu, Malayalam (Limited Working), Italian, Turkish (Elementary)

University Education and Positions

2009 – 2010	Studies of Electronics and Instrumentation Engineering, BIT, India
2010 – 2013	Master of Science Theoretical Physics: Quantum Information Theory, University of Nottingham, Nottingham, UK
2013 – 2016	Master of Science Mathematics/Mathematical Physics: Chern-Simons Theory and Applications, LMU Munich & TU Munich, Munich, Germany
2014 – 2016	Group Member (Gauge/Gravity Duality Group) Max Planck Institute for Physics, Munich, Germany
2016 – 2020	Doctor of Technical Sciences Mathematical & Theoretical Physics: Automorphic forms in String Theory, Technical University of Vienna, Vienna, Austria
2019–	Long term visitor Stanford University, Stanford, CA, United States of America

Grants, honours & awards

2012	David Almond Grant
2019	Marshall Plan Fellowship, endowed by the Austrian Marshall Plan Foundation
2019 – 2020	KuWi Grant of the TU Vienna

Talks and Invited Lectures (Since 2015)

(1) Braneworld Cosmology and 5D Gauge Theories, Munich (Germany), 2015
(2) Black Hole Information and Singularity Resolution, Munich (Germany), 2015
(3) Special Seminar on Black Holes and Quantum Information, MPQ, Munich (Germany), 2015
(4) AdS/CFT and Quantum Error Correction, Munich (Germany), 2015
(5) Microstate counting in Stringy Black Holes and Connections to Number Theory (Invited talk at workshop), Ringberg (Germany), 2015
(6) Firewalls and the Bekenstein Bound as Emergent Quantities from Entanglement, Munich (Germany), 2015
(7) Phase Transitions and Black Hole Instabilities in D = 5 Supergravity, Munich (Germany), 2015
(8) Helical Black Holes in D = 5 SUGRA with Mixed Anomalies (Invited talk at SCGSC conference), London (UK), 2016
(9) Exotic Black Holes in Chern-Simons (super)gravity + Higher Derivative Corrections, Nottingham (UK), Jan 2016
(10) Mass renormalization in bosonic and heterotic string theory, Munich (Germany), 2016
(11) Monster groups and AdS_3 spacetime, Ringberg (Germany), 2016
(12) Some constraints on superconformal field theories, Vienna (Austria), 2017
(13) What *is* wall crossing?, Vienna (Austria), 2017
(14) Topological volume recursion and applications, Vienna (Austria), 2017/18
(15) Calabi-Yau manifolds and sporadic groups, Uni. Wien, Vienna (Austria), 2017
(16) Calabi-Yau manifolds and sporadic groups, IMSc, Chennai (India), 2017
(17) Calabi-Yau manifolds and sporadic groups, IISc, Bangalore (India), 2017
(18) The M_{24} group and elliptic genera of superconformal field theories, Florence (Italy), 2018 (Invited talk at workshop in Florence)
(19) Exact entropy and Rademacher series for CHL orbifolded BPS black holes, Vienna (Austria), 2018 (Invited talk at Moonshine workshop)
(20) $\Gamma_0(N)$, quantum black holes and wall crossing, Lisbon (Portugal), 2018 (Invited seminar speaker)
(21) BPS algebras and moonshine phenomena, Lisbon (Portugal), 2018 (Invited seminar speaker)
(22) Automorphic properties of dyon wall crossing in string theory, Munich (Germany), 2018 (Invited seminar speaker)
(23) Auguries of Physical Mathematics: Colloqium, Vienna (Austria), 2018 (Invited colloquium speaker)
(24) Reconstruction of the Igusa cusp form from localization in supergravity, Vienna (Austria), 2018 (Invited seminar speaker)

(25) DMZ Revisited: Number theory and geometry for negative discriminant states in string theory, Amsterdam (Netherlands), 2019 (Invited seminar speaker)
(26) Comments on number theory and geometry for black holes, Stanford (USA) (Group seminar)
(27) Arithmetic chaos, zeroes of the Riemann $\zeta(s)$ function and topological recursions, Vienna (Austria), (Invited condensed matter theory seminar speaker)
(28) Towards exact matching of black hole degeneracies from localization and number theory, IISc Bangalore, India (Invited seminar speaker), 2019
(29) Towards exact matching of black hole degeneracies from localization and number theory, ICTS Bangalore, India (Invited seminar speaker), 2019
(30) Wall crossing and jumping in string theory and supersymmetric gauge theories, Vienna (Austria), 2019
(31) Counting $\frac{1}{4}-$BPS black holes in $4d, \mathcal{N}=4$ from $\frac{1}{2}-$BPS degeneracies, Stanford (USA), 2019 (Invited seminar speaker)
(32) Single center BH degeneracies in $4d$, $\mathcal{N}=4$ string theory from Dabholkar–Harvey states, ICTP Trieste (Italy), 2020 (Invited seminar speaker)
(33) $\frac{1}{4}-$BPS black holes in $4d, \mathcal{N}=4$ from $\frac{1}{2}-$BPS degeneracies and comments on instanton moduli spaces, Rutgers (USA), 2020 (Invited seminar speaker)

Workshop and conference participation

(1) QFT, Strings and Condensed Matter Physics, Kolymbari (Greece), 2014
(2) Workshop on F Theory, Munich (Germany), 2015
(3) Strings 2015, Bangalore (India), 2015
(4) (Invited) SCGSC, London (UK), 2016
(5) COST-String theory meeting, Milan (Italy), 2017
(6) (Invited) Workshop on AdS_3 and its application, Florence (Italy), 2017
(7) Fields and Duality, Munich (Germany), 2017
(8) (Invited) Supersymmetric Field Theories in the Non-perturbative Regime, Florence (Italy), 2018
(9) Workshop and School on Supersymmetric Localization, Trieste (Italy), 2018
(10) (Invited) Workshop on Moonshine, Vienna (Austria), 2018
(11) (Invited) Simons workshop on Number Theory, Geometry, Moonshine & Strings III, New York (USA), 2019
(12) (Invited) Simons workshop on Automorphic Structures in String Theory, New York (USA), 2019
(13) Strings 2019, Brussels (Belgium), 2019
(14) Workshop on Number Theory and Quantum Physics, Bonn (Germany), 2019

Publications

(1) Chowdhury, A., Kidambi, A., Murthy, S., Reys, V. and Wrase, T., 2019. Dyonic black hole degeneracies in $\mathcal{N}=4$ string theory from Dabholkar-Harvey degeneracies. arXiv preprint arXiv:1912.06562. (Submitted to JHEP)
(2) Banlaki, A., Chattopadhyaya, A., Kidambi, A., Schimannek, T. and Schimpf, M., 2020. Heterotic strings on $(K3\times T^2)/\mathbb{Z}_3$ and their dual Calabi-Yau threefolds. Journal of High Energy Physics, 2020 (1911.09697)

(3) Banlaki, A., Chowdhury, A., Kidambi, A. and Schimpf, M., 2020. On Mathieu moonshine and Gromov-Witten invariants. Journal of High Energy Physics, 2020(2), pp.1-26.
(4) Kidambi, A., Austrian Marshall Plan Fellowship Report: String theoretic realizations of automorphic forms.
(5) Banlaki, A., Chowdhury, A., Kidambi, A., Schimpf, M., Skarke, H. and Wrase, T., 2018. Calabi-Yau manifolds and sporadic groups. Journal of High Energy Physics, 2018(2), p.129.

Teaching

(1) Teaching assistant: Supersymmetry and Supergravity (WiSe 2016-17) (Prof. Peter van Nieuwenhuizen)
(2) Lectures on number theory and stringy black holes (38 hour series), TU Vienna
(3) Seminar series on 'Supersymmetric gauge theories' (WiSe 2017-18)
(4) Seminar series on 'Advanced Supersymmetric gauge theories' (SoSe 18)

Supervision

(1) Mr. Soumil Maulick (Master's thesis): Tunneling in cosmology via Coleman-de Luccia instantons (2019)
(2) Co-supervision of 2 senior projects at Stanford University

References

Dr. Timm Wrase	Prof. Sameer Murthy
TU Wien & Lehigh University	*Kings College London*
timm.wrase@tuwien.ac.at	sameer.murthy@kcl.ac.uk

Prof. Shamit Kachru	Prof. Murat Günaydin
Stanford Unvieristy	*Pennsylvania State University*
skachru@stanford.edu	mgunaydin@psu.edu